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2002 J. Phys. A: Math. Gen. 35 4651

(http://iopscience.iop.org/0305-4470/35/22/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4651–4699 PII: S0305-4470(02)29600-6

A generalization of the Chebyshev polynomials

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Received 2 October 2001, in final form 19 April 2002
Published 24 May 2002
Online at stacks.iop.org/JPhysA/35/4651

Abstract

In this paper we study polynomials that are orthogonal with respect to a weight function which is zero on a set of positive measure. These were initially introduced by Akhiezer as a generalization of the Chebyshev polynomials where the interval of orthogonality is $[-1, \alpha] \cup [\beta, 1]$. Here, this concept is extended and the interval is the union of $g+1$ disjoint intervals, $[-1, \alpha_1] \cup_{j=1}^{g-1} [\beta_j, \alpha_{j+1}] \cup [\beta_g, 1]$, denoted by E.

Starting from a suitably chosen weight function p, and the three-term recurrence relation satisfied by the polynomials, a hyperelliptic Riemann surface is defined, from which we construct representations for both the polynomials of the first (P_n) and second kind (Q_n), respectively, in terms of the Riemann theta function of the surface. Explicit expressions for the recurrence coefficients a_n and b_n are found in terms of theta functions. The second-order ordinary differential equation, where P_n and Q_n/w (where w is the Stieltjes transform of the weight) are linearly independent solutions, is found.

The simpler case, where $g = 1$, is extensively dealt with and the reduction to the Chebyshev polynomials in the limiting situation, $\alpha \to \beta$, where the two intervals merge into one, is demonstrated. We also show that $p(x)k_n(x, x)/n$ for $x \in E$, where $k_n(x, x)$ is the reproducing kernel at coincidence, tends to the equilibrium density of the set E, as $n \to \infty$.

PACS numbers: 02.30.Gp, −2.10.De, 02.30.Mv

1. Preliminaries

A study is made of the generalized Chebyshev polynomials, originally introduced by Akhiezer as a two disjoint interval generalization of the classical Chebyshev polynomials [4]. We extend Akhiezer's original investigation to the case of $g+1$ intervals, where $g > 0$ is an integer (g will turn out to be the genus of a hyperelliptic Riemann surface).

This paper is organized as follows. In this section we collect some classical facts regarding orthogonal polynomials for the convenience of the readers. Those who are familiar with the subject may skip directly to section 2 where the weight function is defined. In section 3 the Riemann surface of a particular hyperelliptic curve is defined. Various facts concerning the surface, including functions and differentials on the surface, are introduced. A theorem about the zeros of certain functions defined on the surface, $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ (which will be of importance in the construction of the polynomials), is also presented. In section 4, after a description of the canonical dissection of the hyperelliptic Riemann surface, the Riemann theta function, with which we obtain explicit expressions for the fundamental objects, $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$, is defined. With the aid of these, we obtain, in section 5, the polynomials of the first and second kind and the associated recurrence coefficients. In section 6, using the Riemann–Roch theorem, we determine differential relations for $P_n(x)$ and $Q_n(x)$ and from these construct a second-order ordinary differential equations with P_n and Q_n/w (w being the Stieltjes transform of the weight p) as linearly independent solutions. Using the results of this section, we give in section 7 an integral representation for the polynomials and consequently deduce the qualitative behaviour of their zeros. The elliptic case ($g = 1$) is studied in detail in section 8. In section 9, we give a description of the determination of the g zeros of the functions $\mathcal{E}_n(x)$ and $\tilde{\mathcal{E}}_n(x)$. The paper concludes with section 10, where we study the large n behaviour of $P_n(x)$ and exhibit the relationship between $k_n(x, x)$ and the equilibrium density of the set E in a certain 'scaling' limit.

As the problem of similar weights has already been posed by Akhiezer, we should like to point out here the difference between our methods and those of Tomchuk [25]. In the paper just mentioned, a more general weight was considered. However, in order to facilitate the construction of the polynomials orthogonal with respect to these weights, an intermediate step was required for the function (3.4), first introduced in [2,3]. This function was expressed, in [25], in terms of quotients of Abelian integrals of the third kind. However, such a construction entails the determination of certain points of the Riemann surface, which was not explicitly found. Indeed, the Jacobi inversion problem which involve these points was not stated. In our formulation (3.4) is expressed in terms of (4.19) together with (5.12), without any unknown parameters. Such a formula made its first appearance in [13], in the context of inverse scattering theory. Note that (4.19), a combination of the Riemann theta functions and a particular Abelian integral of the third kind, appears to be the only way in which an explicit formula can be found without any free parameters. In the recent literature, problems of such a type were also studied [18]; however, more in the vein of [25], namely, with emphasis placed on approximation theory, rather then the explicit constructions of polynomials and the associated recurrence coefficients in terms of theta functions and Abelian integrals. In fact, the author of the first paper cited in [18], stated on p 463, that 'In this paper we do not use elliptic, respectively, Abelian, functions ... ' Our methods clearly differ from those of [18].

Consider the three-term recurrence relations

$$x\mathcal{U}_n = \mathcal{U}_{n+1} + b_{n+1}\mathcal{U}_n + a_n\mathcal{U}_{n-1} \tag{1.1}$$

where $a_n > 0, n = 1, 2, \ldots,$ and $b_n \in \mathbb{R}, n = 0, 1, 2, \ldots$. The two linearly independent polynomial solutions of (1.1) subjected to the initial conditions, $\mathcal{U}_{-1} = 0, \mathcal{U}_0 = 1$ and $\mathcal{U}_0 = 0, \mathcal{U}_1 = 1$ are, respectively,

$$P_n(x) = x^n + p_1x^{n-1} + \cdots, \qquad n \geqslant 1 \tag{1.2}$$

$$Q_n(x) = x^{n-1} + q_1x^{n-2} + \cdots, \qquad n \geqslant 2. \tag{1.3}$$

Note that p_j and q_j, $j = 1, 2, \ldots$ are n dependent. According to a characterization theorem usually attributed to Favard [8], but published in the earlier works of Perron [19], Wintner [29] and Stone [22]: if a_n is strictly positive and b_n real, then P_n satisfies the orthogonality relation with respect to a measure $\mathrm{d}\alpha(x)$:

$$\int_E P_n(x)P_m(x)p(x)\,\mathrm{d}x = h_n\delta_{mn}, \tag{1.4}$$

where p could possibly have mass points. It can be shown that

$$Q_n(x) = \int_E \frac{P_n(x) - P_n(t)}{x - t} p(t)\,\mathrm{d}t. \tag{1.5}$$

By (1.2), taking $p_0 = 1$, we find

$$Q_n(x) = \sum_{j=0}^{n} p_j \sum_{k=0}^{n-1-j} x^k \mu_{n-1-j-k} = x^{n-1} + (\mu_1 + p_1)x^{n-2} + (\mu_2 + p_1\mu_1 + p_2)x^{n-3} + \cdots, \qquad n \geqslant 3, \tag{1.6}$$

where

$$\mu_j := \int_E t^j p(t)\,\mathrm{d}t, \qquad j = 0, 1, 2, \ldots \tag{1.7}$$

are the moments of the weight $p(t)$, with $\mu_0 = 1$. So $q_j = \sum_{k=0}^{j} \mu_{j-k} p_k$.

Regarding the zeros of the polynomials $P_n(x)$ and $Q_n(x)$, if we assume that the weight function $p(x)$, differs only from zero on a union of intervals with $\min_E x = a$ and b, then the following qualitative facts are well known.

Theorem 1.1. *The zeros of $P_n(x)$ are real, simple and contained in the interval (a, b).*

From (1.1) the Christoffel–Darboux identity

$$k_{n+1}(x, t) := \sum_{j=0}^{n} \frac{P_j(x)P_j(t)}{h_j} = \frac{1}{h_n} \frac{P_{n+1}(x)P_n(t) - P_{n+1}(t)P_n(x)}{x - t}, \tag{1.8}$$

can be deduced [23, p 42]. From this result one can deduce the following;

Theorem 1.2. *Between any two zeros of $P_{n+1}(x)$ lies a zero of $P_n(x)$.*

Using the recurrence relation together with the respective initial conditions for $P_n(x)$ and $Q_n(x)$, we have

$$P_n(x)Q_{n+1}(x) - P_{n+1}(x)Q_n(x) = \prod_{j=1}^{n} a_j = h_n. \tag{1.9}$$

From this result we have

Theorem 1.3. *Between consecutive zeros of $P_n(x)$ there is a zero of $Q_n(x)$ and thus the zeros of $Q_n(x)$ are real, simple and contained in (a, b).*

In the following development, the Stieltjes transform of the weight, p, denoted by w, is of some interest:

$$w(x) := \int_E \frac{p(t)}{x - t}\,\mathrm{d}t = \sum_{j=0}^{\infty} \frac{\mu_j}{x^{j+1}}, \qquad x \notin E. \tag{1.10}$$

As $x \to \infty$, we find

$$P_n(x)w(x) - Q_n(x) = \int_E \frac{P_n(t)}{x - t} p(t)\,\mathrm{d}t = \sum_{j=0}^{\infty} \frac{1}{x^{j+1}} \int_E P_n(t)t^j p(t)\,\mathrm{d}t = \frac{1}{x^{n+1}} \int_E P_n(t)t^n p(t)\,\mathrm{d}t + \cdots = \frac{h_n}{x^{n+1}} + \mathrm{O}(x^{-n-2}), \tag{1.11}$$

where the penultimate step above follows from the orthogonality of $\{P_n\}$.

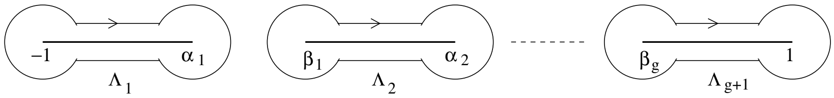

Figure 1. The contour Λ is defined to be the union of the simple loops Λ_j, $j = 1, \ldots, g+1$.

2. Akhiezer's weight function

Throughout this paper we shall denote by E the interval $[-1, \alpha_1] \cup_{j=1}^{g-1} [\beta_j, \alpha_{j+1}] \cup [\beta_g, 1]$ and refer to the complement of E on the interval $[-1, 1]$ as $\bar{E}$. Unless otherwise stated we assume that $-1 < \alpha_j < \beta_j < 1$, $j = 1, \ldots, g$.

Consider the following function defined on the slit complex plane $\mathbb{C}\backslash E$:

$$\tilde{p}(z) = \frac{1}{\pi}\sqrt{\frac{\prod_{j=1}^{g}(z-\alpha_j)}{(z^2-1)\prod_{j=1}^{g}(z-\beta_j)}},$$

where the square root is taken in such a way that $\tilde{p}(z) \sim \frac{1}{\pi z}$, as $\mathrm{Re}\, z \to \infty$. As z tends towards a point on E we obtain two purely imaginary values, each of the same modulus but of opposite sign, depending upon whether the approach is from above or below the cut. This continuation is used to define a weight function p on E. We adopt the convention that, for $x \in E$, $p(x) = \pm \mathrm{i}\tilde{p}(x \pm \mathrm{i}0)$, thus ensuring the weight is positive in E. Hence

$$p(x) := \begin{cases} \dfrac{1}{\pi}\sqrt{\dfrac{\prod_{j=1}^{g}(x-\alpha_j)}{(1-x^2)\prod_{j=1}^{g}(x-\beta_j)}} & \text{for } x \in E \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

This weight function is a generalization of that given in [4, p 158] and is in the form of those considered in [2] and [3].

The Stieltjes transform, w, can be computed easily by contour integration. We apply the Cauchy integral formula to p in the multiply connected domain bounded by a circle of infinite radius and the contour $\Lambda := \cup_{j=1}^{g+1}\Lambda_j$, [17], as shown in figure 1. Note that Λ is sufficiently close to the intervals that make up E, so as to ensure that the point x lies within the multiply connected domain. Then continuously deforming Λ onto the intervals that make up E, we find

$$\begin{aligned} w(x) &:= \int_E \frac{p(t)}{x-t}\,\mathrm{d}t, \qquad x \notin E \\ &= \frac{1}{2}\int_\Lambda \frac{p(t)}{x-t}\,\mathrm{d}t = \sqrt{\frac{\prod_{j=1}^{g}(x-\alpha_j)}{(x^2-1)\prod_{j=1}^{g}(x-\beta_j)}}, \end{aligned} \tag{2.2}$$

where once again we take the branch of the square root that ensures that $w(x)$ is positive for $x > 1$. Expanding $w(x)$ for large $|x|$,

$$w(x) = \frac{1}{x} + \frac{1}{2x^2}\sum_{j=1}^{g}(\beta_j - \alpha_j) + \cdots,$$

from which the first two moments are found to be $\mu_0 = 1$, $\mu_1 = \frac{1}{2}\sum_{j=1}^{g}(\beta_j - \alpha_j)$. Observe that, since

$$P_n(x)w(x) + Q_n(x) = \mathrm{O}(x^{n-1}) \tag{2.3}$$

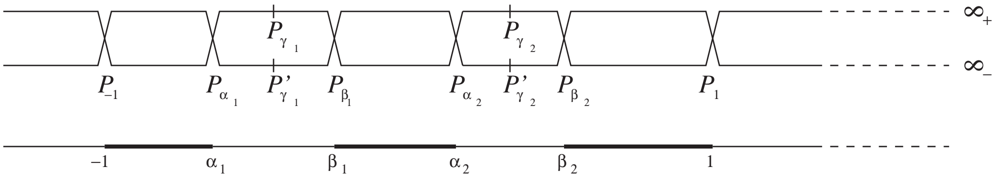

Figure 2. The double sheet structure of $\mathfrak{R}$ for $g = 2$.

it follows from (1.13) that

$$P_n^2(x)w^2(x) - Q_n^2(x) = \mathrm{O}(x^{-2})$$

and consequently

$$P_n^2(x)\prod_{j=1}^{g}(x-\alpha_j) - (x^2-1)Q_n^2(x)\prod_{j=1}^{g}(x-\beta_j) = \eta_g x^g + \cdots + \eta_1 x + \eta_0 =: S_g(x). \tag{2.4}$$

where $\{\eta_j : j = 0, \ldots, g\}$ are determined by the coefficients of P_n and Q_n.

3. The Riemann surface

The compact Riemann surface $\mathfrak{R}$, used throughout this paper is given by the hyperelliptic curve y, where

$$y^2 = (x^2-1)\prod_{j=1}^{g}(x-\alpha_j)(x-\beta_j), \tag{3.1}$$

where g is the genus of the surface. A compact Riemann surface is said to be of genus g if it is conformally equivalent to a sphere with $2g$ holes connected in pairs by g handles.

The double sheet structure of $\mathfrak{R}$, for the case where $g = 2$, is illustrated in figure 2. We refer to the sheet where y is positive (negative) as $x \to \infty$ as the +sheet(−sheet). The point at infinity on these respective sheets is written $\infty_+(\infty_-)$. For any $x \in \mathbb{C}$ there are two points on $\mathfrak{R}$, one on each sheet. If we denote a point on $\mathfrak{R}$ corresponding to some $x \in \mathbb{C}$ as $\mathfrak{p}_x$ then the equivalent point on the other sheet is written $\mathfrak{p}'_x$. If x is a branch point of the hyperelliptic curve then $\mathfrak{p}'_x = \mathfrak{p}_x$. The local parameter around the point $\mathfrak{p}_b$, where b is a branch point, is given by $\xi = \sqrt{x-b}$, while around other points on the surface $\xi = x - b$.

Divisors. A divisor $\mathfrak{D}$ on the Riemann surface $\mathfrak{R}$ is the formal sum of a finite number of its points with integral coefficients:

$$\mathfrak{D} = \sum_j n_j \mathfrak{p}_j, \qquad \mathfrak{p}_j \in \mathfrak{R}. \tag{3.2}$$

The degree of a divisor, denoted by $\deg \mathfrak{D}$, is given by the number $\sum_j n_j$. The set of all divisors on the Riemann surface $\mathfrak{R}$ forms an Abelian group $\mathrm{Div}(\mathfrak{R})$ with respect to the naturally defined operation of addition. We say a divisor $\mathfrak{D}$ is positive and write $\mathfrak{D} \geqslant 0$ if $n_j \geqslant 0$ for all j. This idea can be extended to enable a partial ordering of divisors, taking $\mathfrak{D} \geqslant \mathfrak{D}'$ to mean $\mathfrak{D} - \mathfrak{D}' \geqslant 0$.

The divisor of a given function f on $\mathfrak{R}$, written (f), is defined to be the sum (3.2) where $\mathfrak{p}_i$ is a zero or a pole and n_i the respective multiplicity, as determined by expansions in terms of the local parameter about the relevant point. It is assumed that $n_j > 0$ if $\mathfrak{p}_j$ is a zero and

$n_j < 0$ if $\mathfrak{p}_j$ is a pole. The divisor of an Abelian differential, $(\mathrm{d}\omega)$, is defined in a similar fashion. Note the following results that hold for any function f and any Abelian differential $\mathrm{d}\omega$ defined on a Riemann surface of genus g [21, ch 6], [7, ch 2];

$$\deg(f) = 0, \qquad \deg(\mathrm{d}\omega) = 2g - 2. \tag{3.3}$$

The function f (Abelian differential $\mathrm{d}\omega$) is said to be divisible by the divisor $\mathfrak{D}$ if $(f) - \mathfrak{D} \geqslant 0$ ($(\mathrm{d}\omega) - \mathfrak{D} \geqslant 0$), written as $\mathfrak{D}|f$ ($\mathfrak{D}|\mathrm{d}\omega$). Put another way, if the divisor $\mathfrak{D}$ is

$$\mathfrak{D} = \sum_{j=1}^{l} \mathfrak{p}_{x_j} - \sum_{j=l+1}^{m} \mathfrak{p}_{x_j},$$

then $\mathfrak{D}|f$ means that, counting multiplicities, the points $\{\mathfrak{p}_{x_j} : j = 1, \dots, l\}$ are among the zeros of f and the poles of f are at points from amongst $\{\mathfrak{p}_{x_j} : j = l+1, \dots, m\}$.

Following Akhiezer [4, p 160], consider the function

$$\mathcal{E}_n(\mathfrak{p}_x) := P_n(x) - \frac{Q_n(x)}{w(x)}, \tag{3.4}$$

where w is given by (2.2), but defined on $\mathfrak{R}$. So the range of $\mathcal{E}$ is not confined to the values it takes on the +sheet. In order to determine the divisor composition of $\mathcal{E}_n$, note that

$$\left(\frac{1}{w(x)}\right) = \left(\frac{y}{\prod_{j=1}^{g}(x-\alpha_j)}\right) = -\infty_+ - \infty_- + \mathfrak{p}_1 + \mathfrak{p}_{-1} + \sum_{j=1}^{g}(\mathfrak{p}_{\beta_j} - \mathfrak{p}_{\alpha_j}). \tag{3.5}$$

Near $\infty_\pm$, we find

$$\frac{1}{w(x)} = \pm x + \mathrm{O}(1),$$

so by (1.13) as $\mathfrak{p}_x \to \infty_+$

$$\mathcal{E}_n(\mathfrak{p}_x) = \mathrm{O}(x^{-n}). \tag{3.6}$$

Therefore $\mathcal{E}_n(\mathfrak{p}_x)$ has a zero of order n at ∞_+. As $\mathfrak{p}_x \to \infty_-$, by the definitions of $P_n(x)$ and $Q_n(x)$, we find that for $n > 0$

$$\mathcal{E}_n(\mathfrak{p}_x) = 2x^n + \mathrm{O}(x^{n-1}). \tag{3.7}$$

Therefore $\mathcal{E}_n(\mathfrak{p}_x)$ has a pole of order n at ∞_-. It follows from (3.5) that $\mathcal{E}_n$ has simple poles at $\{\mathfrak{p}_{\alpha_j} : j = 1, \dots, g\}$ and since $\deg(\mathcal{E}_n) = 0$, $\mathcal{E}_n$ must have g zeros at $\{\mathfrak{p}_{\gamma_j} : j = 1, \dots, g\}$. The determination of these zeros, which can in general lie on either sheet of the Riemann surface $\mathfrak{R}$, will be discussed later. For the time being we note that the points $\mathfrak{p}_{\gamma_j}$ depend upon n and the set E. Hence

$$(\mathcal{E}_n(\mathfrak{p}_x)) = n\infty_+ - n\infty_- + \sum_{j=1}^{g}(\mathfrak{p}_{\gamma_j} - \mathfrak{p}_{\alpha_j}). \tag{3.8}$$

For the 'conjugate' function

$$\tilde{\mathcal{E}}_n(\mathfrak{p}_x) := P_n(x) + \frac{Q_n(x)}{w(x)}, \tag{3.9}$$

it can be shown in a similar way that

$$(\tilde{\mathcal{E}}_n(\mathfrak{p}_x)) = -n\infty_+ + n\infty_- + \sum_{j=1}^{g}(\mathfrak{p}'_{\gamma_j} - \mathfrak{p}_{\alpha_j}), \tag{3.10}$$

where it is understood that $\mathfrak{p}'_{\gamma_j}$ lies on the other sheet of $\mathfrak{R}$ to that of $\mathfrak{p}_{\gamma_j}$. Combining this result with (3.8) it follows that

$$(\mathcal{E}_n\tilde{\mathcal{E}}_n) = \sum_{j=1}^{g}(\mathfrak{p}_{\gamma_j} + \mathfrak{p}'_{\gamma_j} - 2\mathfrak{p}_{\alpha_j}). \tag{3.11}$$

Since

$$\mathcal{E}_n(\mathfrak{p}_x)\tilde{\mathcal{E}}_n(\mathfrak{p}_x) = P_n^2(x) - \frac{Q_n^2(x)}{w^2(x)} =: \frac{S_g(x)}{\prod_{j=1}^{g}(x-\alpha_j)}, \tag{3.12}$$

we see that $\mathcal{E}_n\tilde{\mathcal{E}}_n$ has zeros $\{\gamma_j : j = 1, \ldots, g\}$, that satisfy

$$\frac{\eta_j}{\eta_g} = (-1)^{g-j}\varphi_{g-j}(\gamma_1, \ldots, \gamma_g), \qquad j = 0, \ldots, g-1, \tag{3.13}$$

where φ_j is the jth elementary symmetric function and the η_j are as defined in (2.4). We now say that, following qualitative deduction regarding the γ_i:

Theorem 3.1. *The roots of the polynomial $S_g(x)$ are real, simple and when ordered so that $\gamma_1 < \gamma_2 < \cdots < \gamma_g$, are such that $\gamma_j \in [\alpha_j, \beta_j]$, $j = 1, \ldots, g$.*

Proof. The $\{\gamma_j : j = 1, \ldots, g\}$ are the g solutions to the equation

$$S_g(x) = P_n^2(x)\prod_{j=1}^{g}(x-\alpha_j) - Q_n^2(x)(x^2-1)\prod_{j=1}^{g}(x-\beta_j) = 0.$$

Consider any one of the intervals $[\alpha_k, \beta_k]$, $k = 1, \ldots, g$, and note that on the interior points of this interval $(x^2-1)\prod_{j=1}^{g}(x-\beta_j)$ and $\prod_{j=1}^{g}(x-\alpha_j)$ have the same sign. It then follows that $S_g(\alpha_k)$ and $S_g(\beta_k)$ are of opposite sign or that one or possibly both are identically equal to zero. In either case we conclude that the interval contains at least one zero of $S_g(x)$. Now, since $S_g(x)$ is a polynomial of degree g and each of the g intervals $[\alpha_k, \beta_k]$ contains at least one zero, the claim follows. □

The following corollaries will be useful later.

Corollary 3.2. *For any $j = 1, \ldots, g$, $P_n(\beta_j) = 0$ if and only if $\gamma_j = \beta_j$ and $Q_n(\alpha_j) = 0$ if and only if $\gamma_j = \alpha_j$. It then follows that for fixed n we cannot have $P_n(\beta_j) = 0 = Q_n(\alpha_j)$.*

Corollary 3.3. *Given that for a particular j and fixed n, $\gamma_j \in (\alpha_j, \beta_j)$, it follows from the fact that the zeros of $P_n(x)$ are distinct from those of $Q_n(x)$, that γ_j is neither a zero of $P_n(x)$ nor $Q_n(x)$.*

4. The expression of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ in terms of the Riemann theta function

A canonical basis of cycles on $\mathfrak{R}$ is chosen as shown in figure 3 [7, ch 2]. For a hyperelliptic Riemann surface of genus g, a basis for the set of holomorphic differentials is given by

$$\left\{\frac{\mathrm{d}x}{y}, \frac{x\,\mathrm{d}x}{y}, \ldots, \frac{x^{g-1}\,\mathrm{d}x}{y}\right\}.$$

Such differentials that have no pole are called Abelian differentials of the first kind (a differential with pole but vanishing residue is called an Abelian differential of the second kind and a differential with non-vanishing residue is an Abelian differential of the third kind). A normal basis $\{\mathrm{d}\omega_1, \ldots, \mathrm{d}\omega_g\}$ that satisfies the condition

$$\int_{\mathfrak{a}_j} \mathrm{d}\omega_k = \delta_{jk} \qquad j, k = 1, 2, \ldots, g, \tag{4.1}$$

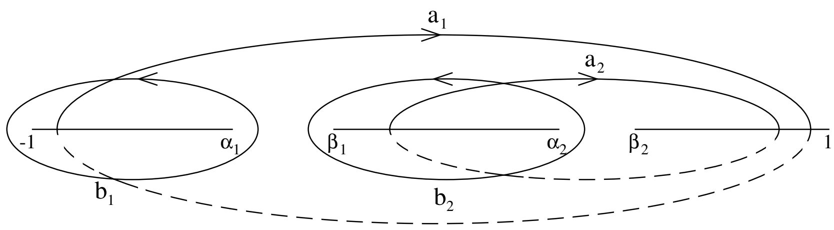

Figure 3. The canonical basis of cycles for $\mathfrak{R}$ with genus g. The parts of the cycles that lie on the − sheet are indicated by broken curves.

is chosen. This is achieved by setting

$$\mathrm{d}\omega_j = \sum_{k=1}^{g} (A^{-1})_{jk} \frac{x^{g-k}\,\mathrm{d}x}{y} \tag{4.2}$$

with the condition that

$$A_{jk} = \int_{\mathrm{a}_k} \frac{x^{g-j}}{y}\,\mathrm{d}x. \tag{4.3}$$

The B-periods matrix of the basis $\{\mathrm{d}\omega_j : j = 1, \ldots, g\}$ is then defined by the relations

$$B_{jk} := \int_{\mathrm{b}_k} \mathrm{d}\omega_j. \tag{4.4}$$

We notice that, for the hyperelliptic surface considered, it is easily shown, by a continuous deformation of the a-cycles, that A is a real matrix. It then follows that $\mathrm{Re}\, B_{jk} = 0$ for $j, k = 1, \ldots, g$. From the Riemann bilinear relationships [7, p 26] this matrix is also symmetric and has the property that $\mathrm{Im}\, B$ is positive definite.

The period lattice Σ is defined by

$$\Sigma := \{\boldsymbol{N} + B\boldsymbol{M} : \boldsymbol{M}, \boldsymbol{N} \in \mathbb{Z}^g\}.$$

If we introduce in $\mathbb{C}^g$ the equivalence relation

$$\boldsymbol{Z} \equiv \boldsymbol{Z}' \Leftrightarrow \boldsymbol{Z} - \boldsymbol{Z}' \in \Sigma,$$

then the Jacobian variety of the Riemann surface $\mathfrak{R}$ is the quotient of $\mathbb{C}^g$ with this equivalence and is written $\mathbb{C}^g/\Sigma$. The Abel map with base point $\mathfrak{p}_1$ maps the surface $\mathfrak{R}$ into it is a Jacobian variety and is given by

$$\boldsymbol{\omega}(\mathfrak{p}) : \mathfrak{R} \ni \mathfrak{p} \longmapsto \int_{\mathfrak{p}_1}^{\mathfrak{p}} \mathrm{d}\boldsymbol{\omega} \in \mathbb{C}^g/\Sigma, \tag{4.5}$$

where

$$\int_{\mathfrak{p}_1}^{\mathfrak{p}} \mathrm{d}\boldsymbol{\omega} := \left(\int_{\mathfrak{p}_1}^{\mathfrak{p}} \mathrm{d}\omega_1, \ldots, \int_{\mathfrak{p}_1}^{\mathfrak{p}} \mathrm{d}\omega_g \right)^T$$

Before proceeding we state two key results from the theory of Riemann surfaces.

Theorem 4.1. (Abel's theorem) *[21, ch 7], [7, ch 2]. The set of points* $\{\mathfrak{p}_j : j = 1, \ldots, m\}$ *and* $\{\mathfrak{q}_j : j = 1, \ldots, m\}$ *are the zeros and poles, respectively, of some meromorphic function on the Riemann surface* $\mathfrak{R}$ *if and only if*

$$\sum_{j=1}^{m} \int_{\mathfrak{q}_j}^{\mathfrak{p}_j} \mathrm{d}\boldsymbol{\omega} \equiv 0 \text{ (modulo the periods)}.$$

Theorem 4.2. (Riemann–Roch theorem) *[21, ch 6], [7, ch 2]. Let the linear space of functions meromorphic on the Riemann surface $\mathfrak{R}$ and divisible by a divisor $\mathfrak{D}$ be $F_{\mathfrak{D}}$ and the space of Abelian differentials on the surface $\mathfrak{R}$ that are divisible by $\mathfrak{D}$ be $\mathrm{d}\Omega_{\mathfrak{D}}$. Then for any divisor $\mathfrak{D}$ on the Riemann surface $\mathfrak{R}$ of genus g,*

$$\dim F_{-\mathfrak{D}} - \dim \mathrm{d}\Omega_{\mathfrak{D}} = 1 - g + \deg \mathfrak{D}.$$

Riemann's theta function ϑ is defined as follows:

$$\vartheta(\boldsymbol{s}; B) = \vartheta(\boldsymbol{s}) := \sum_{\boldsymbol{t}\in\mathbb{Z}^g} \exp(\mathrm{i}\pi(\boldsymbol{t}, B\boldsymbol{t}) + 2\pi\mathrm{i}(\boldsymbol{t}, \boldsymbol{s})), \tag{4.6}$$

where ($\boldsymbol{s} = (\mathrm{s}_1, \ldots, \mathrm{s}_g)^T$) is any complex vector and $(\cdot, \cdot)$ denotes the Euclidean scalar product. The convergence of the theta function is due to the fact that Im B is positive definite. The theta function has the following properties. For $\boldsymbol{t} \in \mathbb{Z}^g$,

$$\begin{array}{ll} \text{symmetry} & \vartheta(-\boldsymbol{s}) = \vartheta(\boldsymbol{s}) \\ \text{periodicity} & (\boldsymbol{s}+\boldsymbol{t}) = \vartheta(\boldsymbol{s}) \\ \text{quasi periodicity} & \vartheta(\boldsymbol{s}+B\boldsymbol{t}) = \mathrm{e}^{-\mathrm{i}\pi[(\boldsymbol{t},B\boldsymbol{t})+2(\boldsymbol{t},\boldsymbol{s})]}\vartheta(\boldsymbol{s}). \end{array} \tag{4.7}$$

We note the following theorem regarding the zeros of theta functions—this result is crucial when representing a meromorphic function on the Riemann surface in terms of theta functions.

Theorem 4.3. *[21, p 167]. If the function*

$$\psi(\mathfrak{p}) := \vartheta(\boldsymbol{\omega}(\mathfrak{p}) - \boldsymbol{s} - \boldsymbol{C}),$$

with $\boldsymbol{C}$ a vector of Riemann constants

$$\mathrm{C}_j = \frac{1 + B_{jj}}{2} - \sum_{1\leqslant k\leqslant g; k\neq j} \int_{\mathrm{a}_k} \omega_j(\mathfrak{p})\, \mathrm{d}\omega_k(\mathfrak{p}), \qquad j = 1, \ldots, g,$$

does not vanish identically in $\mathfrak{p}$, then it has g zeros $q_1, \ldots, q_g$ that satisfy the congruence

$$\sum_{j=1}^{g} \boldsymbol{\omega}(q_j) \equiv \boldsymbol{s}(\text{modulo the periods}).$$

It should be noted that the vector of Riemann constants depends on the base point of the Abel map, which is taken to be $\mathfrak{p}_1$ throughout this paper. Furthermore, for a hyperelliptic Riemann surface, the expression for $\boldsymbol{C}$, can be simplified to

$$\boldsymbol{C} = \sum_{j=1}^{g} \boldsymbol{u}_{\alpha_j}, \tag{4.8}$$

according to [20, theorem 9, p 181]. Using theorem 4.3, together with the fact that, if two meromorphic functions f and g on $\mathfrak{R}$ have the same zeros and poles, then $f = \kappa g$ for some constant κ enables the theta function construction of $\mathcal{E}_n$ [21, p 177]. Using the shorthand notation that $\boldsymbol{u}^\pm := \boldsymbol{\omega}(\infty_\pm)$ and $\boldsymbol{u}_a := \boldsymbol{\omega}(\mathfrak{p}_a)$ for all other points $\mathfrak{p}_a$ on the surface $\mathfrak{R}$, from (3.8) we write

$$\mathcal{E}_n(\mathfrak{p}_x) = \delta_n \frac{\vartheta^n(\boldsymbol{u}_x - \boldsymbol{u}^+ + \boldsymbol{K}) \prod_{j=1}^{g} \vartheta(\boldsymbol{u}_x - \boldsymbol{u}_{\gamma_j} + \boldsymbol{K})}{\vartheta^n(\boldsymbol{u}_x - \boldsymbol{u}^- + \boldsymbol{K}) \prod_{j=1}^{g} \vartheta(\boldsymbol{u}_x - \boldsymbol{u}_{\alpha_j} + \boldsymbol{K})}, \tag{4.9}$$

where $\boldsymbol{K} := -\boldsymbol{C} - \sum_{j=2}^{g} \boldsymbol{u}_{\beta_j}$ and for any given n, δ_n is a constant that depends upon E only, the value of which is easily determined by the behaviour of $\mathcal{E}_n$ at a point on $\mathfrak{R}$. Note the choice of $\{\mathfrak{p}_{\beta_j} : j = 2, \ldots, g\}$ as the other $g-1$ zeros of the respective theta functions in the construction above. This choice is not unique, but since the divisor $\sum_{j=2}^{g} \mathfrak{p}_{\beta_j}$ is *non-special* or *general* it is a choice that ensures that none of the composite theta functions are identically zero. For further information on general divisors we refer the reader to [21, ch 10] and [7, p 33], where the former reference presents a complete account on the zeros of Riemann's theta function and explains in particular the conditions under which a theta function is identically zero. We point the reader to theorem 3 on p 169.

Clearly the vectors $\boldsymbol{u}_{\gamma_j}$ depend upon the $\mathfrak{p}_{\gamma_j}$ which in turn depends upon n. The n dependence can be uncovered straightforwardly from Abel's theorem:

$$\sum_{j=1}^{g} \boldsymbol{u}_{\gamma_j} \equiv \sum_{j=1}^{g} \boldsymbol{u}_{\alpha_j} - n(\boldsymbol{u}^{+} - \boldsymbol{u}^{-})(\text{modulo the periods}). \tag{4.10}$$

The determination of $\{\mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$ from (4.10) is known as the Jacobi inversion problem [21, ch 8], the solution to which is that the $\{\mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$ are the g zeros of the function $\vartheta(\boldsymbol{u}_x - \boldsymbol{C}_0)$ where

$$\boldsymbol{C}_0 = \boldsymbol{C} + \sum_{j=1}^{g} \boldsymbol{u}_{\alpha_j} - n(\boldsymbol{u}^{+} - \boldsymbol{u}^{-}).$$

Note that there is in fact a stronger congruence relating the $\{\mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$ to known points on the surface. Writing $\boldsymbol{u}_x \to \boldsymbol{u}_x + B\boldsymbol{e}_k$, where $\boldsymbol{e}_k$ is a g vector defined by $(\boldsymbol{e}_k)_l = \delta_{kl}$, $k, l = 1, \ldots, g$, we require that this transformation leaves the expression for $\mathcal{E}_n(\mathfrak{p}_x)$ unchanged. It then follows from the quasi-periodic property of the theta function given in relationship (4.7) that

$$\sum_{j=1}^{g} \boldsymbol{u}_{\gamma_j} \equiv \sum_{j=1}^{g} \boldsymbol{u}_{\alpha_j} - n(\boldsymbol{u}^{+} - \boldsymbol{u}^{-}) \ (\text{mod } \mathbb{Z}^g). \tag{4.11}$$

The following expression for $\tilde{\mathcal{E}}_n(\mathfrak{p}_x)$ is found in a completely analogous manner to that above: using (3.10) we find

$$\tilde{\mathcal{E}}_n = \tilde{\delta}_n \frac{\vartheta^n(\boldsymbol{u}_x - \boldsymbol{u}^{-} + \boldsymbol{K}) \prod_{j=1}^{g} \vartheta(\boldsymbol{u}_x - \boldsymbol{u}'_{\gamma_j} + \boldsymbol{K})}{\vartheta^n(\boldsymbol{u}_x - \boldsymbol{u}^{+} + \boldsymbol{K}) \prod_{j=1}^{g} \vartheta(\boldsymbol{u}_x - \boldsymbol{u}_{\alpha_j} + \boldsymbol{K})}, \tag{4.12}$$

where we have written $\boldsymbol{u}'_{\gamma_j} := \boldsymbol{\omega}(\mathfrak{p}'_{\gamma_j})$. Again using the condition that this expression is invariant under the transformation $\boldsymbol{u}_x \to \boldsymbol{u}_x + B\boldsymbol{e}_k$, we uncover a relationship between the $\{\mathfrak{p}'_{\gamma_j} : j = 1, \ldots, g\}$ and known points on the surface, finding that

$$\sum_{j=1}^{g} \boldsymbol{u}'_{\gamma_j} \equiv \sum_{j=1}^{g} \boldsymbol{u}_{\alpha_j} + n(\boldsymbol{u}^{+} - \boldsymbol{u}^{-}) \ (\text{mod } \mathbb{Z}^g). \tag{4.13}$$

It is also possible to give an alternative representation of the functions $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$. Consider the Abelian integral of the third kind:

$$\Omega(\mathfrak{p}_x) = \int_{\mathfrak{p}_1}^{\mathfrak{p}_x} \mathrm{d}\Omega := \int_{\mathfrak{p}_1}^{\mathfrak{p}_x} \frac{t^g + \sum_{j=0}^{g-1} k_j t^j}{y} \, \mathrm{d}t. \tag{4.14}$$

This integral is normalized [7, p 29] in such a way that

$$\int_{\mathfrak{a}_k} \mathrm{d}\Omega = 0, \qquad k = 1, \ldots, g, \tag{4.15}$$

thus providing g equations for the determination of $\{k_j : j = 0, \ldots, g-1\}$. Note that this requirement implies that

$$\int_{\alpha_k}^{\beta_k} \mathrm{d}\Omega = 0, \qquad k = 1, \ldots, g. \tag{4.16}$$

We denote by $\hat{\boldsymbol{B}}$ the vector whose components are

$$\hat{\mathrm{B}}_j = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{b}_j} \mathrm{d}\Omega, \qquad j = 1, \ldots, g. \tag{4.17}$$

Observe that, as $\mathfrak{p} \to \infty_\pm$,

$$\Omega(\mathfrak{p}_x) = \pm \ln x + \mathrm{O}(1). \tag{4.18}$$

This fact and theorem 4.3 lead to the following representations [7, section 2.7]:

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \Delta_n \mathrm{e}^{-n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\boldsymbol{u}_x - \sum_{j=1}^g \boldsymbol{u}_{\gamma_j} - \boldsymbol{C})}{\vartheta(\boldsymbol{u}_x - \sum_{j=1}^g \boldsymbol{u}_{\alpha_j} - \boldsymbol{C})} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \tilde{\Delta}_n \mathrm{e}^{n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\boldsymbol{u}_x - \sum_{j=1}^g \boldsymbol{u}'_{\gamma_j} - \boldsymbol{C})}{\vartheta(\boldsymbol{u}_x - \sum_{j=1}^g \boldsymbol{u}_{\alpha_j} - \boldsymbol{C})}, \end{aligned} \tag{4.19}$$

where Δ_n and $\tilde{\Delta}_n$ depend upon E and n alone and can be determined, as we shall see in the next section, by knowledge of the local behaviour of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ around certain points on the Riemann surface. Note that the path joining the points $\mathfrak{p}_1$ and $\mathfrak{p}_x$ is the same in both $\Omega(\mathfrak{p}_x)$ and $\boldsymbol{u}_x$. If we now add to this contour an arbitrary cycle, b_k, $k = 1, \ldots, g$, the requirement that both $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ be invariant under such an alteration to the path of integration then implies the following relationships:

$$\begin{aligned} \sum_{j=1}^g \boldsymbol{u}_{\gamma_j} &\equiv \sum_{j=1}^g \boldsymbol{u}_{\alpha_j} - n\hat{\boldsymbol{B}} \ (\mathrm{mod}\ \mathbb{Z}^g) \\ \sum_{j=1}^g \boldsymbol{u}'_{\gamma_j} &\equiv \sum_{j=1}^g \boldsymbol{u}_{\alpha_j} + n\hat{\boldsymbol{B}} \ (\mathrm{mod}\ \mathbb{Z}^g). \end{aligned} \tag{4.20}$$

Thus observing (4.11), we note the following congruence:

$$\hat{\boldsymbol{B}} \equiv \boldsymbol{u}^+ - \boldsymbol{u}^- \ (\mathrm{mod}\ \mathbb{Z}^g). \tag{4.21}$$

Henceforth we denote $\boldsymbol{D} := 2\sum_{j=1}^g \boldsymbol{u}_{\alpha_j}$, consequently referring to (4.8),

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \Delta_n \mathrm{e}^{-n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\boldsymbol{u}_x + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}_x - \boldsymbol{D})} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \tilde{\Delta}_n \mathrm{e}^{n\Omega(\mathfrak{p}_x)} \frac{\vartheta(\boldsymbol{u}_x - n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}_x - \boldsymbol{D})}. \end{aligned} \tag{4.22}$$

The advantage of these representations over those of (4.9) and (4.12) is that the functions $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are given explicitly without needing to solve the Jacobi inversion problem. In this respect, we note that this is a particularly useful feature when considering the general scenario where $g > 1$. In the elliptic case where $g = 1$ both types of expression are equally applicable. These explicit expressions for $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are similar to those which first appeared in the theory of integrable systems. In this context this type of expression was first put forward in [13] and applied in [13–15] to the construction of finite-gap solutions of the non-linear Schrödinger and KdV equations.

5. The polynomials P_n and Q_n and the coefficients of the recurrence relation they satisfy

In this section the importance of the theta function constructions for $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ is realized. Using these expressions we determine explicit formulae for the polynomials of the first and second kind that follow from consideration of the Akhiezer weight. In the process we shall determine expressions for the coefficients of the recurrence relation these polynomials satisfy.

From (1.13) observe that, as $\mathfrak{p}_x \to \infty_+$,

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{1}{w(x)} \sum_{j=n}^{\infty} \frac{1}{x^{j+1}} \int_E P_n(t) t^j p(t)\, dt,$$

where $w(x) = x^{-n} + O(x^{-n-1})$. Writing $t^n = \sum_{k=0}^{n} \lambda_k(n) P_{n-k}(t)$, where $\lambda_0(n) = 1$, and using the orthogonality relation satisfied by P_n, we find

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{h_n}{x^n} + O(x^{-n-1}), \qquad \mathfrak{p}_x \to \infty_+. \tag{5.1}$$

It also follows from (1.13) that for $n \geqslant 1$ as $\mathfrak{p}_x \to \infty_-$, $P_n(x) = -\frac{Q_n(x)}{w(x)} + O(x^{-n})$ and hence

$$\mathcal{E}_n(\mathfrak{p}_x) = 2P_n(x) + O(x^{-n}), \qquad \mathfrak{p}_x \to \infty_- (n \geqslant 1). \tag{5.2}$$

We can then similarly deduce that

$$\tilde{\mathcal{E}}_n(\mathfrak{p}_x) = \begin{cases} \dfrac{h_n}{x^n} + O(x^{-n-1}) & \text{as } \mathfrak{p}_x \to \infty_- \\ 2P_n(x) + O(x^{-n}) & \text{as } \mathfrak{p}_x \to \infty_+ \ (n \geqslant 1). \end{cases} \tag{5.3}$$

Now we compare the results above with expansions obtained from (4.22). Examining the behaviour of $\Omega(\mathfrak{p}_x)$ as $\mathfrak{p}_x \to \infty_\pm$, recall that

$$\Omega(\mathfrak{p}_x) = \int_{\mathfrak{p}_1}^{\mathfrak{p}_x} \frac{t^g + \sum_{j=0}^{g-1} k_j t^j}{y}\, dt, \tag{5.4}$$

where we assume without loss of generality that the path of integration projects onto the interval $[1, x]$. Writing $s = \frac{1}{x}$, we find that, as $\mathfrak{p}_x \to \infty_\pm$,

$$\frac{d\Omega}{ds} = \mp\left[\frac{1}{s} + \left(k_{g-1} + \frac{1}{2}\sum_{j=1}^{g}(\alpha_j + \beta_j)\right) + O(s)\right].$$

Thus it follows that

$$\Omega(\mathfrak{p}_x) = \begin{cases} \ln x + \chi_0 - \dfrac{\chi_1}{x} + O(x^{-2}) & \text{as } \mathfrak{p}_x \to \infty_+ \\ -\ln x - \chi_0 + \dfrac{\chi_1}{x} + O(x^{-2}) & \text{as } \mathfrak{p}_x \to \infty_-, \end{cases} \tag{5.5}$$

where

$$\begin{aligned} \chi_0 &= \int_{\mathfrak{p}_1}^{\infty_+} \left(\frac{t^g + \sum_{j=0}^{g-1} k_j t^j}{y} - \frac{1}{t}\right) dt \\ \chi_1 &= k_{g-1} + \tfrac{1}{2}\sum_{j=1}^{g}(\alpha_j + \beta_j). \end{aligned} \tag{5.6}$$

Considering the behaviour of the g vector $\boldsymbol{u}_x$ in such limits we write $\boldsymbol{u}_x = \boldsymbol{u}^\pm + \delta\boldsymbol{u}$. Hence it follows that

$$\delta\boldsymbol{u} = \int_{\infty_\pm}^{\mathfrak{p}_x} d\boldsymbol{\omega}, \tag{5.7}$$

where from (4.2)

$$d\omega_j = \sum_{k=1}^{g} (A^{-1})_{jk} \frac{x^{g-k}\, dx}{y}, \qquad j = 1, \ldots, g.$$

It is then easily shown that

$$\delta u_j = \mp\frac{(A^{-1})_{j1}}{x} + O(x^{-2}), \qquad \mathfrak{p}_x \to \infty_\pm. \tag{5.8}$$

Note the following series expansion:

$$\vartheta(\boldsymbol{u}+\delta\boldsymbol{u}) = \vartheta(\boldsymbol{u}) + \sum_{j=1}^{g} \vartheta'_j(\boldsymbol{u})\delta \mathrm{u}_j + \mathrm{O}(\delta\mathrm{u}_j\delta\mathrm{u}_k) \tag{5.9}$$

where

$$\vartheta'_j(\boldsymbol{u}) := \frac{\partial}{\partial \mathrm{u}_j}\vartheta(\boldsymbol{u}).$$

Thus expanding (4.22) around the point at ∞_+,

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \frac{\Delta_n \mathrm{e}^{-n\chi_0}\vartheta(\boldsymbol{u}^+ + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})} x^{-n} + \mathrm{O}(x^{-n-1}) \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \frac{\tilde{\Delta}_n \mathrm{e}^{n\chi_0}\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})} x^{n} + \mathrm{O}(x^{n-1}). \end{aligned} \tag{5.10}$$

Similarly around ∞_-,

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \frac{\Delta_n \mathrm{e}^{n\chi_0}\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})} x^{n} + \mathrm{O}(x^{n-1}) \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \frac{\tilde{\Delta}_n \mathrm{e}^{-n\chi_0}\vartheta(\boldsymbol{u}^+ + n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})} x^{-n} + \mathrm{O}(x^{-n-1}), \end{aligned} \tag{5.11}$$

where we have observed that $\boldsymbol{u}^\pm \in \mathbb{R}^g$, which follows from the reality of A, and consequently implies that $\boldsymbol{u}^+ + \boldsymbol{u}^- \equiv 0$ modulo $\mathbb{Z}^g$. Comparing the ∞_- expression for $\mathcal{E}_n$ and the ∞_+ expression for $\tilde{\mathcal{E}}_n$ with the respective results of (5.2) and (5.3), we deduce that for $n \geqslant 1$

$$\begin{aligned} \Delta_n &= 2\mathrm{e}^{-n\chi_0}\frac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})} \\ \tilde{\Delta}_n &= 2\mathrm{e}^{-n\chi_0}\frac{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})}. \end{aligned} \tag{5.12}$$

Using this result, it then follows by equating the first expression of (5.10) and the second in (5.11) with their respective expansions given in (5.1) and (5.3) that, for $n \geqslant 1$,

$$\begin{aligned} h_n &= 2\mathrm{e}^{-2n\chi_0}\frac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})\vartheta(\boldsymbol{u}^+ + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})} \\ &= 2\mathrm{e}^{-2n\chi_0}\frac{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(\boldsymbol{u}^+ + n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})}. \end{aligned} \tag{5.13}$$

It follows straightforwardly from the recurrence relation of (1.1) and the orthogonality relationship satisfied by P_n that $a_n h_{n-1} = h_n$, so that using the first result of (5.13),

$$a_n = \begin{cases} 2\mathrm{e}^{-2\chi_0}\dfrac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})\vartheta(\boldsymbol{u}^+ + \hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(\boldsymbol{u}^+ - \hat{\boldsymbol{B}} + \boldsymbol{D})} & \text{if } n = 1 \\ \mathrm{e}^{-2\chi_0}\dfrac{\vartheta(\boldsymbol{u}^+ - (n-1)\hat{\boldsymbol{B}} + \boldsymbol{D})\vartheta(\boldsymbol{u}^+ + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ + (n-1)\hat{\boldsymbol{B}} - \boldsymbol{D})\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})} & \text{if } n > 1, \end{cases} \tag{5.14}$$

since $h_0 = 1$.

In order to determine an expression for the other recurrence coefficient b_n, we expand the expression for $\mathcal{E}_n$ given in (4.22) about the point at ∞_-, obtaining the first two terms. Using (5.5) and (5.9) it follows, after some elementary calculations, that

$$\mathcal{E}_n(\mathfrak{p}_x) = 2x^n + 2(\mathcal{K}(n) - \mathcal{K}(0) - n\chi_1)x^{n-1} + \mathrm{O}(x^{n-2}), \qquad \mathfrak{p}_x \to \infty_-, \tag{5.15}$$

where

$$\mathcal{K}(n) := \sum_{j=1}^{g} (A^{-1})_{j1} \frac{\vartheta_j'(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})}. \tag{5.16}$$

Thus by using (5.2) we identify

$$p_1(n) = \mathcal{K}(n) - \mathcal{K}(0) - n\chi_1. \tag{5.17}$$

From the relation (1.1) it is easily seen that $b_n = p_1(n-1) - p_1(n)$. Consequently

$$b_n = \mathcal{K}(n-1) - \mathcal{K}(n) + \chi_1. \tag{5.18}$$

It is possible to make this expression more explicit; from the orthogonality relationship of (1.4) it can be shown that

$$b_1 = -p_1(n) = \tfrac{1}{2} \sum_{j=1}^{g} (\beta_j - \alpha_j).$$

Thus comparing this with the expression obtained by setting $n = 1$ in (5.18) we find

$$\chi_1 = \tfrac{1}{2} \sum_{j=1}^{g} (\beta_j - \alpha_j) + \mathcal{K}(1) - \mathcal{K}(0),$$

and consequently that

$$\begin{aligned} b_n &= \tfrac{1}{2} \sum_{j=1}^{g} (\beta_j - \alpha_j) + \mathcal{K}(1) - \mathcal{K}(0) + \mathcal{K}(n-1) - \mathcal{K}(n) \\ &= \tfrac{1}{2} \textstyle\sum_{j=1}^{g} (\beta_j - \alpha_j) + \sum_{j=1}^{g} (A^{-1})_{j1} \left[\frac{\vartheta_j'(\boldsymbol{u}^+ - \hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \hat{\boldsymbol{B}} + \boldsymbol{D})} - \frac{\vartheta_j'(\boldsymbol{u}^+ + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})} + \frac{\vartheta_j'(\boldsymbol{u}^+ - (n-1)\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - (n-1)\hat{\boldsymbol{B}} + \boldsymbol{D})} - \frac{\vartheta_j'(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})} \right]. \end{aligned} \tag{5.19}$$

It is shown in section 10 that $\boldsymbol{D}$ is given by (10.33). Using this, the quasi-periodicity of the theta function and the reality of $\boldsymbol{u}^{\pm}$ and $\hat{\boldsymbol{B}}$, we see that a_n given by (5.14) is strictly positive and b_n given by (5.19) is real.

Having determined Δ_n and $\tilde{\Delta}_n$, from (4.22) we have explicit expressions for $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$. The polynomials are

$$\begin{aligned} P_n(x) &= \tfrac{1}{2}(\mathcal{E}_n(\mathfrak{p}_x) + \tilde{\mathcal{E}}_n(\mathfrak{p}_x)) \\ Q_n(x) &= \frac{w(x)}{2}(\tilde{\mathcal{E}}_n(\mathfrak{p}_x) - \mathcal{E}_n(\mathfrak{p}_x)). \end{aligned} \tag{5.20}$$

In terms of the theta functions, for $n \geqslant 1$,

$$\begin{aligned} P_n(x) = {} & \mathrm{e}^{-n(\chi_0 + \Omega(\mathfrak{p}_x))} \frac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})\vartheta(\boldsymbol{u}_x + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})\vartheta(\boldsymbol{u}_x - \boldsymbol{D})} \\ & + \mathrm{e}^{n(\Omega(\mathfrak{p}_x) - \chi_0)} \frac{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(\boldsymbol{u}_x - n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})\vartheta(\boldsymbol{u}_x - \boldsymbol{D})}, \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} Q_n(x) = {} & \sqrt{\frac{\prod_{j=1}^{g}(x - \alpha_j)}{(x^2 - 1)\prod_{j=1}^{g}(x - \beta_j)}} \left[\mathrm{e}^{n(\Omega(\mathfrak{p}_x) - \chi_0)} \frac{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(\boldsymbol{u}_x - n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})\vartheta(\boldsymbol{u}_x - \boldsymbol{D})} \right. \\ & \left. - \mathrm{e}^{-n(\chi_0 + \Omega(\mathfrak{p}_x))} \frac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})\vartheta(\boldsymbol{u}_x + n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} + \boldsymbol{D})\vartheta(\boldsymbol{u}_x - \boldsymbol{D})} \right], \end{aligned} \tag{5.22}$$

where in both these expressions $\mathfrak{p}_x$ can, without loss of generality, be taken to lie on the +sheet of $\mathfrak{R}$, with the path of integration from $\mathfrak{p}_1$ to $\mathfrak{p}_x$ the same for both $\boldsymbol{u}_x$ and $\Omega(\mathfrak{p}_x)$.

Another interesting quantity in the general theory of orthogonal polynomials, called the moment or Hankel matrix, is denoted as $\mathcal{H}_n$ and defined to be

$$\mathcal{H}_n = [\mu_{j+k}]_{j,k=0,1,\ldots,n}, \tag{5.23}$$

where the μ_j are the moments of a given weight function. According to [23, p 28] the determinant of this matrix is given by $\prod_{j=1}^{n} h_j$. Consequently, using (5.13), we find that the Hankel determinant for the weight considered here is given by

$$\det \mathcal{H}_n = 2^n \exp[-n(n+1)\chi_0] \left[\frac{\vartheta(\boldsymbol{u}^+ + \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})} \right]^n \frac{\vartheta((2n+1)\boldsymbol{u}^+ - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})}, \tag{5.24}$$

where we have noticed that (4.21) implies $\hat{\boldsymbol{B}} \equiv 2\boldsymbol{u}^+$ modulo $\mathbb{Z}^g$.

6. The second-order differential equation satisfied by $P_n(x)$

In this section we concern ourselves with the determination of various differential relations satisfied by the polynomials. In particular, by using the Riemann–Roch theorem we devise representations for $(\mathrm{d}/\mathrm{d}x) \ln \mathcal{E}_n$ and $(\mathrm{d}/\mathrm{d}x) \ln \tilde{\mathcal{E}}_n$ in terms of algebraic functions of x. When compared with equivalent expressions obtained directly from the definitions of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$, we can ultimately derive a second-order ordinary differential equation satisfied by P_n and Q_n/w.

We proceed by investigating the divisor structure of $(\mathrm{d}/\mathrm{d}x) \ln \mathcal{E}_n$ and $(\mathrm{d}/\mathrm{d}x) \ln \tilde{\mathcal{E}}_n$. It has already been shown that, as $\mathfrak{p}_x \to \infty_+$[1],

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &\sim x^{-n} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &\sim x^n. \end{aligned} \tag{6.1}$$

In the neighbourhood of ∞_-

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &\sim x^{n} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &\sim x^{-n}. \end{aligned} \tag{6.2}$$

Locally around $\mathfrak{p}_{\alpha_j}$, $j = 1, \ldots, g$, it follows from (5.9) that

$$\vartheta(\boldsymbol{u}_x - \boldsymbol{D}) \simeq \sum_{k=1}^{g} \vartheta_k'(\boldsymbol{u}_{\alpha_j} - \boldsymbol{D}) \delta \mathrm{u}_k,$$

where

$$\delta \mathrm{u}_k = \int_{\mathfrak{p}_{\alpha_j}}^{\mathfrak{p}_x} \mathrm{d}\omega_k.$$

Since α_j is a branch point of the hyperelliptic curve that defines the surface $\mathfrak{R}$, the local parameter is given by $\xi = \sqrt{x - \alpha_j}$. Using (4.2) we can expand the integrands in terms of ξ, finding that $\delta \mathrm{u}_k = \mathrm{O}(\xi)$. Hence

$$\vartheta(\boldsymbol{u}_x - \boldsymbol{D}) \sim \xi, \quad \mathfrak{p}_x \to \mathfrak{p}_{\alpha_j},$$

and thus in this limit it follows from (4.19) that

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &\sim \xi^{-1} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &\sim \xi^{-1}. \end{aligned} \tag{6.3}$$

[1] Note that throughout this paper we use the following convention. The relation $f(x) \simeq g(x)$ as $x \to x_0$, means that $\lim_{x\to x_0} f(x)/g(x) = 1$ and $f(x) \sim g(x)$ as $x \to x_0$ means there exist positive constants A and B such that $A < \lim_{x\to x_0} |f(x)/g(x)| < B$.

Allowing $\mathfrak{p}_x \to \mathfrak{p}_{\gamma_j}$, $j = 1, \ldots, g$,

$$\vartheta(\boldsymbol{u}_x + n\hat{\boldsymbol{B}} - \boldsymbol{D}) \simeq \sum_{k=1}^{g} \vartheta'_k(\boldsymbol{u}_{\gamma_j} + n\hat{\boldsymbol{B}} - \boldsymbol{D})\delta \mathrm{u}_k,$$

where

$$\delta \mathrm{u}_k = \int_{\mathfrak{p}_{\gamma_j}}^{\mathfrak{p}_x} \mathrm{d}\omega_k.$$

We assume without loss of generality that $\gamma_j \in (\alpha_j, \beta_j)$ (the reader will note that the singular cases where $\gamma_j = \alpha_j$ or β_j follow quite naturally later by allowing γ_j to tend towards either of these points). Hence the local parameter is $\xi = x - \gamma_j$. Expansion of the integrand then gives $\delta \mathrm{u}_j = \mathrm{O}(\xi)$ and consequently

$$\mathcal{E}_n(\mathfrak{p}_x) \sim \xi, \qquad \mathfrak{p}_x \to \mathfrak{p}_{\gamma_j}. \tag{6.4}$$

Similarly we find that, as $\mathfrak{p}_x \to \mathfrak{p}'_{\gamma_j}$, $j = 1, \ldots, g$, $\xi = x - \gamma_j$ and

$$\tilde{\mathcal{E}}_n(\mathfrak{p}_x) \sim \xi. \tag{6.5}$$

Using the above results, we are now in a position to determine the behaviour of $\mathrm{d}\ln\mathcal{E}_n/\mathrm{d}x$ and $\mathrm{d}\ln\tilde{\mathcal{E}}_n/\mathrm{d}x$ in the locality of all the points where they diverge. In the following $\mathcal{P}_0(x)$ denotes a regular power series in x with non-zero coefficient of x^0. Using (4.19), we summarize the findings.

Around ∞_+:

$$\begin{aligned} \mathcal{E}_n &= \frac{1}{x^n}\mathcal{P}_0\left(\frac{1}{x}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = -\frac{n}{x} + \mathrm{O}(x^{-2}). \\ \tilde{\mathcal{E}}_n &= x^n\mathcal{P}_0\left(\frac{1}{x}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\tilde{\mathcal{E}}_n = \frac{n}{x} + \mathrm{O}(x^{-2}). \end{aligned} \tag{6.6}$$

Around ∞_-:

$$\begin{aligned} \mathcal{E}_n &= x^n\mathcal{P}_0\left(\frac{1}{x}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = \frac{n}{x} + \mathrm{O}(x^{-2}). \\ \tilde{\mathcal{E}}_n &= \frac{1}{x^n}\mathcal{P}_0\left(\frac{1}{x}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\tilde{\mathcal{E}}_n = -\frac{n}{x} + \mathrm{O}(x^{-2}). \end{aligned} \tag{6.7}$$

Around $\mathfrak{p}_{\alpha_j}$ ($\xi = \sqrt{x - \alpha_j}$):

$$\begin{aligned} \mathcal{E}_n &= \frac{1}{\xi}\mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = -\frac{1}{2\xi^2} + \mathrm{O}(\xi^{-1}). \\ \tilde{\mathcal{E}}_n &= \frac{1}{\xi}\mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = -\frac{1}{2\xi^2} + \mathrm{O}(\xi^{-1}). \end{aligned} \tag{6.8}$$

Around $\mathfrak{p}_{\gamma_j}$ ($\xi = x - \gamma_j$):

$$\mathcal{E}_n = \xi\mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = \frac{1}{\xi} + \mathrm{O}(1). \tag{6.9}$$

Around $\mathfrak{p}'_{\gamma_j}$ ($\xi = x - \gamma_j$):

$$\tilde{\mathcal{E}}_n = \xi\mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\tilde{\mathcal{E}}_n = \frac{1}{\xi} + \mathrm{O}(1). \tag{6.10}$$

Around $\mathfrak{p}_b = \mathfrak{p}_{\pm 1}, \mathfrak{p}_{\beta_1}$ or $\mathfrak{p}_{\beta_2}$ $(\xi = \sqrt{x-b})$:

$$\begin{aligned} \mathcal{E}_n &= \mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \ln \tilde{\mathcal{E}}_n \sim \frac{1}{\xi}. \\ \tilde{\mathcal{E}}_n &= \mathcal{P}_0(\xi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \ln \tilde{\mathcal{E}}_n \sim \frac{1}{\xi}. \end{aligned} \tag{6.11}$$

It is easily verified from (4.19) that locally around all other points on $\mathfrak{R}$ the functions $(\mathrm{d}/\mathrm{d}x)\ln \mathcal{E}_n$ and $(\mathrm{d}/\mathrm{d}x)\ln \tilde{\mathcal{E}}_n$ are bounded. It then becomes clear that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \ln \mathcal{E}_n\right) = \infty_+ + \infty_- - \mathfrak{p}_1 - \mathfrak{p}_{-1} - \sum_{j=1}^{g} (2\mathfrak{p}_{\alpha_j} + \mathfrak{p}_{\beta_j} + \mathfrak{p}_{\gamma_j}) + 4g \text{ zeros} =: -\mathfrak{d} + 4g \text{ zeros,} \tag{6.12}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \ln \tilde{\mathcal{E}}_n\right) = \infty_+ + \infty_- - \mathfrak{p}_1 - \mathfrak{p}_{-1} - \sum_{j=1}^{g} (2\mathfrak{p}_{\alpha_j} + \mathfrak{p}_{\beta_j} + \mathfrak{p}'_{\gamma_j}) + 4g \text{ zeros} =: -\tilde{\mathfrak{d}} + 4g \text{ zeros.} \tag{6.13}$$

Having determined the divisor structure, we now find representations for $(\mathrm{d}/\mathrm{d}x)\ln \mathcal{E}_n(\mathfrak{p}_x)$ and $(\mathrm{d}/\mathrm{d}x)\ln \tilde{\mathcal{E}}_n(\mathfrak{p}_x)$ in terms of algebraic functions on the Riemann surface. We denote the linear space of meromorphic functions on $\mathfrak{R}$ that are divisible by $-\mathfrak{d}$ as $F_{-\mathfrak{d}}$. Recall that this is the space of functions that include at least simple zeros at ∞_+ and ∞_- amongst their zeros and which only have poles at points from amongst the set $\{\mathfrak{p}_1, \mathfrak{p}_{-1}, \mathfrak{p}_{\alpha_j}, \mathfrak{p}_{\beta_j}, \mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$, provided that poles at the $\mathfrak{p}_{\alpha_i}$ are no more than double poles and those at other points from the set are simple poles.

Theorem 6.1. *With the divisor* $\mathfrak{d}$ *defined by (6.12), the dimension of the linear space* $F_{-\mathfrak{d}}$ *is* $3g + 1$.

Proof. Suppose that there exists $\mathrm{d}\Omega$, a differential on $\mathfrak{R}$ with the property that $\mathfrak{d}|\mathrm{d}\Omega$. This requires that $\mathrm{d}\Omega$ has at least simple zeros at the points $\mathfrak{p}_{-1}$, $\mathfrak{p}_1$, $\{\mathfrak{p}_{\beta_j} : j = 1, \ldots, g\}$ and $\{\mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$ and double zeros at the points $\{\mathfrak{p}_{\alpha_j} : j = 1, \ldots, g\}$, and having no more than simple poles at ∞_+ and ∞_- as they are only poles. This implies that such a differential would have the property that $\deg(\mathrm{d}\Omega) \geqslant 4g$. However, by virtue of (3.3), $\deg(\mathrm{d}\Omega) = 2g - 2$, a contradiction that allows us to conclude that there is no Abelian differential on $\mathfrak{R}$ that is divisible by $\mathfrak{d}$. It then follows directly from the Riemann–Roch theorem that $\dim F_{-\mathfrak{d}} = 3g + 1$. □

It may be shown in an entirely analogous way that $\dim F_{-\tilde{\mathfrak{d}}} = 3g + 1$. We can now construct a basis for the set of meromorphic functions on $\mathfrak{R}$ that are divisible by $-\mathfrak{d}$. It is easily verified that the following set is suitable:

$$\left\{\frac{1}{y}, \frac{x}{y}, \ldots, \frac{x^g}{y}, \frac{1}{x-\alpha_1}, \ldots, \frac{1}{x-\alpha_g}, \frac{y+y_1}{y(x-\gamma_1)}, \ldots, \frac{y+y_g}{y(x-\gamma_g)}\right\},$$

where, for $j = 1, \ldots, g$,

$$y_i := y|_{\mathfrak{p}_x = \mathfrak{p}_{\gamma_j}}. \tag{6.14}$$

Similarly a basis for $F_{-\tilde{\mathfrak{d}}}$ is given by

$$\left\{\frac{1}{y}, \frac{x}{y}, \ldots, \frac{x^g}{y}, \frac{1}{x-\alpha_1}, \ldots, \frac{1}{x-\alpha_g}, \frac{y-y_1}{y(x-\gamma_1)}, \ldots, \frac{y-y_g}{y(x-\gamma_g)}\right\}.$$

It is now possible to represent both $(\mathrm{d}/\mathrm{d}x)\ln\mathcal{E}_n(\mathfrak{p}_x)$ and $(\mathrm{d}/\mathrm{d}x)\ln\tilde{\mathcal{E}}_n(\mathfrak{p}_x)$ as a linear combination of the elements of these respective basis sets, writing

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = \frac{1}{y}\sum_{j=0}^{g} c_j x^j + \sum_{j=1}^{g}\frac{c_{g+j}}{x-\alpha_j} + \sum_{j=1}^{g} c_{2g+j}\frac{y+y_j}{y(x-\gamma_j)} \tag{6.15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\tilde{\mathcal{E}}_n = \frac{1}{y}\sum_{j=0}^{g} \tilde{c}_j x^j + \sum_{j=1}^{g}\frac{\tilde{c}_{g+j}}{x-\alpha_j} + \sum_{j=1}^{g} \tilde{c}_{2g+j}\frac{y-y_j}{y(x-\gamma_j)}, \tag{6.16}$$

where the sets of coefficients $\{c_j : j = 0,\ldots,3g\}$ and $\{\tilde{c}_j : j = 0,\ldots,3g\}$ remain to be determined.

Using the results (6.8)–(6.10) to equate leading coefficients from the expansions of the right-hand sides of (6.15) and (6.16) around $\mathfrak{p}_{\alpha_j}$, $\mathfrak{p}_{\gamma_j}$ and $\mathfrak{p}'_{\gamma_j}$,respectively, then gives

$$\begin{aligned} c_{g+j} &= \tilde{c}_{g+j} = -\tfrac{1}{2}, \qquad & j &= 1,\ldots,g \\ c_{2g+j} &= \tilde{c}_{2g+j} = \tfrac{1}{2}, \qquad & j &= 1,\ldots,g, \end{aligned} \tag{6.17}$$

while equating leading coefficients for the expansions around $\infty_\pm$ gives

$$\begin{aligned} &c_g + \sum_{j=1}^{g} c_{g+j} + \sum_{j=1}^{g} c_{2g+j} = -n \\ &\tilde{c}_g + \sum_{j=1}^{g} \tilde{c}_{g+j} + \sum_{j=1}^{g} \tilde{c}_{2g+j} = n, \end{aligned} \tag{6.18}$$

and thus $-c_g = \tilde{c}_g = n$.

In order to determine expressions for the other coefficients in (6.15) and (6.16) further terms in the expansions of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are required. Allowing $\mathfrak{p}_x \to \infty_-$, since $w(x) = \mathrm{O}(x^{-1})$, we have by virtue of (1.13) that

$$\mathcal{E}_n(\mathfrak{p}_x) = 2P_n(x) + \mathrm{O}(x^{-n}). \tag{6.19}$$

Hence we may write

$$\mathcal{E}_n(\mathfrak{p}_x) = 2x^n\sum_{j=0}^{n} p_j x^{-j} + x^n\sum_{j=0}^{\infty} r_j x^{-2n-j} = x^n\sum_{j=0}^{\infty}\mathsf{a}_j x^{-j}, \tag{6.20}$$

where

$$\mathsf{a}_j := \begin{cases} 2p_j & \text{for } 0 \leqslant j \leqslant n \\ 0 & \text{for } n+1 \leqslant j \leqslant 2n-1 \\ r_{j-2n} & \text{for } j \geqslant 2n. \end{cases} \tag{6.21}$$

Consequently we find that

$$\mathcal{E}_n'(x) = x^{n-1}\sum_{j=0}^{\infty}(n-j)\mathsf{a}_j x^{-j}.$$

We also require the expansion of y as $\mathfrak{p}_x \to \infty_-$:

$$y = -x^{g+1}\sum_{j=0}^{\infty}\mathsf{b}_j x^{-j}, \tag{6.22}$$

where the coefficients, b_j, are easily determined. From (6.15), we note that

$$\mathcal{E}_n' y + \mathcal{E}_n y \frac{1}{2} \sum_{j=1}^{g} \left(\frac{1}{x-\alpha_j} - \frac{1}{x-\gamma_j} - \frac{y_j}{y(x-\gamma_j)} \right) = \mathcal{E}_n \sum_{j=0}^{g} c_j x^j.$$

Expanding around the point at ∞_- throughout and multiplying the various series representations together, we find that

$$-\sum_{j=0}^{g} (\mathsf{d}_j + \mathsf{f}_j) x^{-j} = \sum_{j=0}^{g} \mathsf{g}_j x^{-j} + \mathrm{O}(x^{-1-g}),$$

with

$$\begin{aligned} \mathsf{d}_j &:= \sum_{k=0}^{i} (n+k-j) \mathsf{b}_j \mathsf{a}_{i-j} \\ \mathsf{f}_j &:= \tfrac{1}{2} \sum_{k=0}^{i} \sum_{l=1}^{g} \sum_{m=0}^{j-k} (\alpha_l^k - \gamma_l^k) \mathsf{b}_m \mathsf{a}_{j-k-m} \\ \mathsf{g}_j &:= \sum_{k=0}^{j} \mathsf{a}_k c_{g-j+k}. \end{aligned} \tag{6.23}$$

Equating coefficients of x^{-j}, we obtain a sequence of equations from which we can, in principle, iteratively determine the c_j, from $j = g$ to 0, in terms of $\{\mathsf{a}_k : k = 0, \ldots, g-j\}$ and $\{\mathsf{b}_k : k = 0, \ldots, g-j\}$, namely

$$-\mathsf{d}_j - \mathsf{f}_j = \mathsf{g}_j, \qquad j = 0, \ldots, g. \tag{6.24}$$

In particular, note that when $g = 1$, since $c_1 = -n$, $\mathsf{b}_0 = 1$, $\mathsf{b}_1 = -(\alpha_1 + \beta_1)/2$, $\mathsf{a}_0 = 2$ and $\mathsf{a}_1 = 2p_1$, it follows that, for $n > 0$,

$$c_0 = p_1 + n \frac{\alpha_1 + \beta_1}{2} + \frac{\gamma_1 - \alpha_1}{2}. \tag{6.25}$$

It follows from (1.13) that, as $\mathfrak{p}_x \to \infty_+$,

$$\tilde{\mathcal{E}}_n(\mathfrak{p}_x) = 2P_n(x) + \mathrm{O}(x^{-n}).$$

In this limit we also have

$$y = x^{g+1} \sum_{j=0}^{\infty} \mathsf{b}_j x^{-j}.$$

From (6.16), we find

$$\tilde{\mathcal{E}}_n' y + \tilde{\mathcal{E}}_n y \frac{1}{2} \sum_{j=1}^{g} \left(\frac{1}{x-\alpha_j} - \frac{1}{x-\gamma_j} + \frac{y_j}{y(x-\gamma_j)} \right) = \tilde{\mathcal{E}}_n \sum_{j=0}^{g} \tilde{c}_j x^j,$$

so that expanding this expression around the point at ∞_+ and equating coefficients, it becomes apparent in the light of the preceding calculations that $\tilde{c}_j = -c_j$ for $j = 0, \ldots, g$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \mathcal{E}_n = -\frac{n x^g}{y} + \sum_{j=0}^{g-1} \frac{c_j x^j}{y} - \frac{1}{2} \sum_{j=1}^{g} \frac{1}{x-\alpha_j} + \frac{1}{2} \sum_{j=1}^{g} \frac{y + y_j}{y(x-\gamma_j)} \tag{6.26}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \tilde{\mathcal{E}}_n = \frac{n x^g}{y} - \sum_{j=0}^{g-1} \frac{c_j x^j}{y} - \frac{1}{2} \sum_{j=1}^{g} \frac{1}{x-\alpha_j} + \frac{1}{2} \sum_{j=1}^{g} \frac{y - y_j}{y(x-\gamma_j)}. \tag{6.27}$$

Introducing the function

$$\Pi(x) := \prod_{j=1}^{g}(x-\alpha_i),$$

it follows from the respective definitions (3.4) and (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n = \frac{\Pi^2 P_n' - (y'Q_n + yQ_n')\Pi + \Pi' y Q_n}{\Pi(\Pi P_n - yQ_n)} \tag{6.28}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\tilde{\mathcal{E}}_n = \frac{\Pi^2 P_n' + (y'Q_n + yQ_n')\Pi - \Pi' y Q_n}{\Pi(\Pi P_n + yQ_n)}. \tag{6.29}$$

Equating (6.26) with (6.28) and (6.27) with (6.29) provides

$$y(\Pi^2 P_n' - (y'Q_n + yQ_n')\Pi - \Pi' y Q_n)$$
$$= \left[-nx^g + \sum_{j=0}^{g-1} c_j x^j - \frac{y}{2}\sum_{j=1}^{g}\frac{1}{x-\alpha_j} + \frac{1}{2}\sum_{j=1}^{g}\frac{y+y_j}{x-\gamma_j}\right]\Pi(\Pi P_n - yQ_n) \tag{6.30}$$

and

$$y(\Pi^2 P_n' + (y'Q_n + yQ_n')\Pi - \Pi' y Q_n)$$
$$= \left[nx^g - \sum_{j=0}^{g-1} c_j x^j - \frac{y}{2}\sum_{j=1}^{g}\frac{1}{x-\alpha_j} + \frac{1}{2}\sum_{j=1}^{g}\frac{y-y_j}{x-\gamma_j}\right]\Pi(\Pi P_n + yQ_n) \tag{6.31}$$

The addition of (6.30) to (6.31) gives

$$P_n'(x) = f_1(x)P_n(x) + f_2(x)Q_n(x) \tag{6.32}$$

with

$$f_1(x) := \frac{1}{2}\sum_{j=1}^{g}\left(\frac{1}{x-\gamma_j} - \frac{1}{x-\alpha_j}\right) \tag{6.33}$$

and

$$f_2(x) := \frac{nx^g - \sum_{j=0}^{g-1} c_j x^j - \frac{1}{2}\sum_{j=1}^{g}\frac{y_j}{x-\gamma_j}}{\prod_{j=1}^{g}(x-\alpha_j)}, \tag{6.34}$$

and a subtraction of (6.30) from (6.31) yields

$$Q_n'(x) = f_3(x)P_n(x) + f_4(x)Q_n(x) \tag{6.35}$$

with

$$f_3(x) := \frac{nx^g - \sum_{j=0}^{g-1} c_j x^j - \frac{1}{2}\sum_{j=1}^{g}\frac{y_j}{x-\gamma_j}}{(x^2-1)\prod_{j=1}^{g}(x-\beta_j)} \tag{6.36}$$

and

$$f_4(x) := -\frac{x}{x^2-1} + \frac{1}{2}\sum_{j=1}^{g}\left(\frac{1}{x-\gamma_j} - \frac{1}{x-\beta_j}\right). \tag{6.37}$$

Recalling

$$w(x) := \sqrt{\frac{\prod_{j=1}^{g}(x-\alpha_j)}{(x^2-1)\prod_{j=1}^{g}(x-\beta_j)}},$$

we observe the following relations between the $f_j(x)$:

$$f_3(x) = w^2(x) f_2(x) \tag{6.38}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln w(x) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x^2-1) + \sum_{j=1}^{g}[\ln(x-\beta_j) + \ln(x-\alpha_j)]\right) = f_4(x) - f_1(x). \tag{6.39}$$

Introducing $R_n := Q_n/w$, (6.35) implies that

$$R_n'(x) = \frac{f_3(x)}{w(x)} P_n(x) + \left(f_4(x) - \frac{w'(x)}{w(x)}\right) R_n(x).$$

This equation, rewritten using (6.38) and (6.39), together with the recast form of (6.32) then gives the following coupled system of equations:

$$\begin{aligned} P_n'(x) &= f_1(x) P_n(x) + f_2(x) w(x) R_n(x) \\ R_n'(x) &= f_2(x) w(x) P_n(x) + f_1(x) R_n(x). \end{aligned} \tag{6.40}$$

The differential equations satisfied by $P_n(x)$ and $R_n(x)$ are simply obtained by elimination of the other function from the system, the symmetry of which indicates that these equations must be identical. Hence we find that $P_n(x)$ and $R_n(x)$ are the linearly independent solutions to

$$Y''(x) - \left(2f_1 + \frac{f_2'}{f_2} + \frac{w'}{w}\right) Y'(x) + \left(f_1^2 - f_1' + f_1\left(\frac{f_2'}{f_2} + \frac{w'}{w}\right) - f_2^2 w^2\right) Y(x) = 0 \tag{6.41}$$

7. Alternative representations for $P_n(x)$ and $Q_n(x)$

The coupled system of differential equations given in (6.40) enables the determination of alternative representations for both $P_n(x)$ and $Q_n(x)$ for real values of x. As we shall see these new identities have the advantageous property, when compared to the previous expressions of section 5, that the oscillatory structure is explicitly apparent. It is this feature that allows us to make further inferences regarding the location of the zeros of both polynomials.

By setting $\tilde{R}_n(x) := -\mathrm{i}R_n(x)$ and $\tilde{w}(x) = \mathrm{i}w(x)$ we rewrite (6.40) in a more convenient way, obtaining

$$\begin{aligned} P_n'(x) &= f_1(x) P_n(x) + f_2(x) \tilde{w}(x) \tilde{R}_n(x) \\ \tilde{R}_n'(x) &= -f_2(x) \tilde{w}(x) P_n(x) + f_1(x) \tilde{R}_n(x). \end{aligned} \tag{7.1}$$

Using the Prüfer substitution [26], where

$$\begin{aligned} P_n(x) &= \rho_n(x) \cos\theta_n(x) \\ \tilde{R}_n(x) &= \rho_n(x) \sin\theta_n(x), \end{aligned} \tag{7.2}$$

the equations of (7.1) are decoupled, giving

$$\begin{aligned} (\ln \rho_n(x))' &= f_1(x) \\ \theta_n'(x) &= -f_2(x) \tilde{w}(x). \end{aligned} \tag{7.3}$$

Integrating the first equation along a smooth non-self-intersecting contour that lies entirely in the upper half of the complex plane, between 1 and an arbitrary point z that satisfies the condition $\operatorname{Im} z > 0$, we obtain

$$\rho_n(z) = \rho_n(1) \sqrt{\prod_{j=1}^{g} \frac{(z-\gamma_j)(1-\alpha_j)}{(z-\alpha_j)(1-\gamma_j)}}. \tag{7.4}$$

The second equation then gives

$$\theta_n(z) - \theta_n(1) = \int_1^z \frac{\mathrm{i}\left(\sum_{j=0}^{g-1} c_j t^j - nt^g + \frac{1}{2}\sum_{j=1}^{g} \frac{y_j}{t-\gamma_j}\right)}{y}\,\mathrm{d}t, \tag{7.5}$$

where once again z is taken to be an arbitrary complex point with the property that $\mathrm{Im}\, z > 0$ and the path of integration is a contour as above. For the purpose of our investigation we also assume, without loss of generality, that the positive branch of y is selected (i.e. integration on the Riemann surface of y is along a contour between 1 and $\mathfrak{p}_z$ that lies entirely in the upper half of the +sheet).

From (7.2), observe that

$$\rho_n^2(x) = P_n^2(x) + \tilde{R}_n^2(x) = P_n(x)^2 - \frac{Q_n^2(x)}{w^2(x)} \tag{7.6}$$

and

$$\tan\theta_n(x) = \frac{\tilde{R}_n(x)}{P_n(x)} = \frac{Q_n(x)}{\tilde{w}(x)P_n(x)}. \tag{7.7}$$

It then follows that

$$\begin{aligned} &\rho_n(1) = \pm P_n(1) \\ &\tan\theta_n(1) = 0, \end{aligned} \tag{7.8}$$

which then implies the following representations for $P_n(z)$ and $Q_n(z)$ valid for $\mathrm{Im}\, z > 0$:

$$\begin{aligned} P_n(z) &= P_n(1)\sqrt{\prod_{j=1}^{g}\frac{(z-\gamma_j)(1-\alpha_j)}{(z-\alpha_j)(1-\gamma_j)}}\cos\Psi_n(z) \\ Q_n(z) &= \mathrm{i}\frac{P_n(1)}{\sqrt{z^2-1}}\sqrt{\prod_{j=1}^{g}\frac{(z-\gamma_j)(1-\alpha_j)}{(z-\beta_j)(1-\gamma_j)}}\sin\Psi_n(z), \end{aligned} \tag{7.9}$$

where the function $\Psi_n(z)$ is given by

$$\Psi_n(z) = \int_1^z \theta_n'(t)\,\mathrm{d}t = \mathrm{i}\int_1^z \frac{\left(\sum_{j=0}^{g-1} c_j t^j - nt^g + \frac{1}{2}\sum_{j=1}^{g}\frac{y_j}{t-\gamma_j}\right)}{y}\,\mathrm{d}t, \tag{7.10}$$

the path of integration being as previously described. Notice that the sign of $\rho_n(1)$ is selected so that $P_n(z) \to P_n(1)$ as $z \to 1$. Expressions for the polynomials for real values are then obtained by the analytical continuation of those above, allowing $z \to x \in \mathbb{R}$. The reader should note that it can be verified that these expressions fulfil the necessary criterion of being analytic at the end points of the intervals that make up E. It should also be noted that, although the formulae of (7.9) have the benefit of exposing the oscillatory behaviour of the polynomials, they should not be seen as a replacement to the representations given in (5.21) and (5.22). This is because, in order to make the expressions of (7.9) completely explicit as those results of section 5 are, we require knowledge about the y_j, $j = 1, \ldots, g$, and this necessitates the solution of the Jacobi inversion problem of (4.10).

We proceed by investigating the behaviour of the function $\Psi_n(x)$ on the intervals that make up $\bar{E}$. From (7.7) it follows that, for $j = 1, \ldots, g$,

$$\tan\,\theta_n(\beta_j) = \begin{cases} \pm\infty & \text{if } P_n(\beta_j) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.11}$$

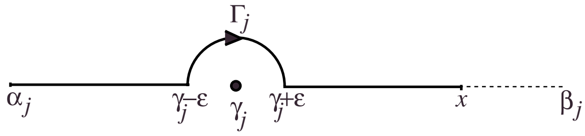

Figure 4. The contour Γ_j used in determining $\Psi_n(x)$ when $x \in (\gamma_j, \beta_j)$.

and

$$\tan\theta_n(\alpha_j) = \begin{cases} 0 & \text{if } Q_n(\alpha_j) = 0 \\ \pm\infty & \text{otherwise.} \end{cases} \tag{7.12}$$

In the light of corollary 3.2, for a fixed n we can identify three distinct eventualities that must be dealt with separately. Namely, these are the general case where $\gamma_j \in (\alpha_j, \beta_j)$ and the two singular cases where either $Q_n(\alpha_j) = 0$, so that $\gamma_j = \alpha_j$, or $P_n(\beta_j) = 0$, implying that $\gamma_j = \beta_j$. We deal with the general case first, discussing the behaviour of $\Psi_n(x)$ on an arbitrary interval (α_j, β_j) for a fixed n such that $\gamma_j \in (\alpha_j, \beta_j)$. In such instances the path of integration in the expression for $\Psi_n(x)$ can be continuously deformed so that it consists of two sections, one joining the points 1 and α_j and a second from α_j to the point $x \in (\alpha_j, \beta_j)$. Since $\tan\theta_n(\alpha_j) = \pm\infty$, we can certainly write

$$\Psi_n(x) = (l_{\alpha_j} - \tfrac{1}{2})\pi + \int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t, \tag{7.13}$$

where l_{α_j} is an integer that depends upon n and the set E, while the path of integration joining α_j and x is smooth and non-self-intersecting, lying in the upper half of the complex plane so that, if $x > \gamma_j$, it passes above γ_j. When $x \in (\alpha_j, \gamma_j)$ this contour may be deformed so that it lies along the real axis. However, when $x \in (\gamma_j, \beta_j)$ we deform the path onto the contour Γ_j, illustrated in figure 4. By allowing $\varepsilon \to 0$ in this contour, only the pole at γ_j in the integrand of $\Psi_n(x)$ contributes to the integral over the semi-circular arc of Γ_j. Consequently we find the following result for $\Psi_n(x)$:

$$\Psi_n(x) = \begin{cases} (l_{\alpha_j} - \tfrac{1}{2})\pi + \displaystyle\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t & \text{for } x \in (\alpha_j, \gamma_j) \\ (l_{\alpha_j} - \tfrac{1}{2})\pi \pm \dfrac{\pi}{2} + P\displaystyle\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t & \text{for } x \in (\gamma_j, \beta_j), \end{cases} \tag{7.14}$$

where in the second case the P indicates that the integral is considered in the principal value sense and the plus sign is selected if $\mathfrak{p}_{\gamma_j}$ lies on the +sheet of $\mathfrak{R}$, the minus sign if it lies on the −sheet. The reader should note that this last result may be used to verify that the expressions for the polynomials are analytic at γ_j.

We now consider the behaviour of $\Psi_n(x)$ on the general interval (α_j, β_j), for n such that $\gamma_j = \alpha_j$. Since it follows from corollary 3.2 that $P_n(\beta_j) \neq 0$, by (7.11) $\tan\theta_n(\beta_j) = 0$. Hence, by deforming the integration contour of $\Psi_n(x)$, so that it consists of an arc in the upper half of the complex plane from 1 and β_j, and a section along the real axis from β_j to the point x, we may write

$$\Psi_n(x) = l_{\beta_j}\pi + \int_{\beta_j}^{x} \theta_n'(t)\,\mathrm{d}t, \qquad x \in (\alpha_j, \beta_j), \tag{7.15}$$

where $l_{\beta_j} \in \mathbb{Z}$ depends upon n and the set E.

Finally, if $\gamma_j = \beta_j$ for a particular n, it follows from corollary 3.2 that $Q_n(\alpha_j) \neq 0$. From (7.12), we then find that $\tan\theta_n(\alpha_j) = \pm\infty$. Thus deforming the integration path of $\Psi_n(x)$, so that it has a section lying on the real axis between α_j and $x \in (\alpha_j, \beta_j)$, gives

$$\Psi_n(x) = (l_{\alpha_j} - \tfrac{1}{2})\pi + \int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t, \qquad x \in (\alpha_j, \beta_j), \tag{7.16}$$

where $l_{\alpha_i} \in \mathbb{Z}$. Note that, since l_{α_i} depends upon n, in general we would expect it to take a different value here to that in (7.13).

Having considered all the possible types of behaviour of $\Psi_n(x)$ for $x \in \bar{E}$, we are in a position to prove the following theorem regarding the zeros of $P_n(x)$ and $Q_n(x)$ in this open set:

Theorem 7.1. *The interval (α_j, β_j), $j = 1, \ldots, g$, contains at most one zero of each of the polynomials $P_n(x)$ and $Q_n(x)$. Further to this, if we consider a particular interval (α_j, β_j), in cases where n is such that $\gamma_j \in (\alpha_j, \beta_j)$, any zero of $P_n(x)$ on this interval must lie in (α_j, γ_j) and any zero of $Q_n(x)$ is contained in (γ_j, β_j). If n is such that $\gamma_j = \alpha_j$ or $\gamma_j = \beta_j$, then both $P_n(x)$ and $Q_n(x)$ have no zeros on the interval.*

Proof. Taking the general interval (α_j, β_j) we notice that, at all points on this interval, θ_n' has the property that $\mathrm{Re}\{\theta_n'\} = 0$, regardless of n. Considering the general case where $\gamma_j \in (\alpha_j, \beta_j)$ it then follows from (7.14) that

$$|\cos\Psi_n(x)| = \begin{cases} \left|\sinh\left(\mathrm{Im}\left[\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right)\right| & \text{for } x \in (\alpha_j, \gamma_j) \\ \cosh\left(\mathrm{Im}\left[P\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right) & \text{for } x \in (\gamma_j, \beta_j), \end{cases}$$

whilst

$$|\sin\Psi_n(x)| = \begin{cases} \cosh\left(\mathrm{Im}\left[\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right) & \text{for } x \in (\alpha_j, \gamma_j) \\ \left|\sinh\left(\mathrm{Im}\left[P\int_{\alpha_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right)\right| & \text{for } x \in (\gamma_j, \beta_j). \end{cases}$$

From corollary 3.3, note that $P_n(\gamma_j) \neq 0$ and $Q_n(\gamma_j) \neq 0$, so that if $P_n(x)$ has zeros on (α_j, β_j) they must lie on the interval (α_j, γ_j) and any zeros of $Q_n(x)$ lying on (α_j, β_j) must be contained in (γ_j, β_j). Suppose then that $P_n(x)$ has two or more zeros on (α_j, γ_j). Since $Q_n(x)$ has a zero between any two consecutive zeros of $P_n(x)$ by theorem 1.3, it follows that $Q_n(x)$ must have a zero on (α_j, γ_j), a contradiction that forces us to conclude that $P_n(x)$ has at most one zero on this interval. Similar arguments show that $Q_n(x)$ has at most one zero on the interval (γ_j, β_j).

If $Q_n(\alpha_j) = 0$ for a particular n, so that $\gamma_j = \alpha_j$, then from (7.15) we deduce that

$$|\cos\Psi_n(x)| = \cosh\left(\mathrm{Im}\left[\int_{\beta_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right), \qquad x \in (\alpha_j, \beta_j).$$

This implies that $P_n(x) \neq 0$ when $x \in (\alpha_j, \beta_j)$. Since $Q_n(\alpha_j) = 0$, to prevent contradicting theorem 1.3, we must conclude that $Q_n(x)$ has no zeros on (α_j, β_j) also.

In cases where n is such that $P_n(\beta_j) = 0$ and thus $\gamma_j = \beta_j$, it follows from (7.16) that

$$|\sin\Psi_n(x)| = \cosh\left(\mathrm{Im}\left[\int_{\beta_j}^{x} \theta_n'(t)\,\mathrm{d}t\right]\right), \qquad x \in (\alpha_j, \beta_j),$$

implying that $Q_n(x) \neq 0$ for $x \in (\alpha_j, \beta_j)$. Because $P_n(\beta_j) = 0$, theorem 1.3 demands that $P_n(x) \neq 0$ on this interval too. □

8. Explicit consideration of the elliptic problem

The elliptic problem refers to the case where $E = [-1, \alpha] \cup [\beta, 1]$ and the Riemann surface we work on is that which is defined by the curve $y^2 = (x^2 - 1)(x - \alpha)(x - \beta)$. For an account of Akhiezer's work on this problem we refer the reader to [4, ch 10]. The genus 1 case merits explicit study since it is the simplest possible generalization of the classical one-interval problem and this transparency permits an easier contrast between the different behaviours than consideration of the general genus g.

Before progressing we draw attention to an alteration in the notation. For $g = 1$ the Riemann theta function of previous sections is a function of a single variable, and throughout this section we identify $\vartheta(u)$ with $\vartheta_3(u)$ in accordance with the notation of Jacobi. It is also convenient to introduce here three other Jacobian theta functions:

$$\begin{aligned}\vartheta_1(u) &:= 2\sum_{j=0}^{\infty}(-1)^j q^{(j+\frac{1}{2})^2}\sin(2j+1)\pi u\\ \vartheta_2(u) &:= 2\sum_{j=0}^{\infty} q^{(j+\frac{1}{2})^2}\cos(2j+1)\pi u\\ \vartheta_3(u) &:= 1+2\sum_{j=1}^{\infty} q^{j^2}\cos 2j\pi u\\ \vartheta_4(u) &:= 1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2}\cos 2j\pi u,\end{aligned} \tag{8.1}$$

where the quantity $q := \exp[\mathrm{i}\pi\tau]$, with τ the single element of the B period matrix. An excellent review on the properties of these functions can be found in [28, ch 9].

The polynomials. In the genus 1 case various quantities can be given in terms of elliptic integrals. The following integrals are commonly defined (see, for instance, [12]):

$$A_{jk} := A = 2\int_{\mathfrak{p}_\beta}^{\mathfrak{p}_\alpha}\frac{\mathrm{d}x}{y} = \frac{4K}{\sqrt{(1-\alpha)(1+\beta)}}, \tag{8.2}$$

where the integration is performed along a path in the +sheet of $\mathfrak{R}$ that projects onto a section of the real axis $[\beta, \alpha]$. $K(k)$ is a complete elliptic integral of the first kind, with the modulus k given by

$$k := \sqrt{\frac{2(\beta-\alpha)}{(1-\alpha)(1+\beta)}}. \tag{8.3}$$

The elements of the B period matrix are given by

$$B_{jk} := \tau = \frac{2}{A}\int_{\mathfrak{p}_{-1}}^{\mathfrak{p}_\alpha}\frac{\mathrm{d}x}{y} = \mathrm{i}\frac{K'}{K}, \tag{8.4}$$

where the integration is along the upper edge of the branch cut in the +sheet of $\mathfrak{R}$. The function $K' := K(k')$ is a complete elliptic integral of the first kind with a modulus k' complementary to k:

$$k' := \sqrt{1-k^2} = \sqrt{\frac{(1-\beta)(1+\alpha)}{(1-\alpha)(1+\beta)}}. \tag{8.5}$$

As A is real it follows that $\boldsymbol{u}^{\pm} = \mathrm{u}^{\pm} \in \mathbb{R}$. Also by considering the function $x - 1$ on $\mathfrak{R}$, using Abel's theorem we find that $\boldsymbol{u}^{+} + \boldsymbol{u}^{-} \equiv 0$ (modulo the periods). Together these facts imply that

$\boldsymbol{u}^- = \mathrm{u}^- = 1 - \mathrm{u}^+$, where u^+ is the value of the Abelian integral in the fundamental period parallelogram:

$$\mathrm{u}^+ = \frac{1}{A}\int_{\mathfrak{p}_1}^{\infty_+} \frac{\mathrm{d}x}{y} = \frac{1}{2}\frac{\mathrm{F}(\phi, k)}{K}, \tag{8.6}$$

where the integration is along a path in the +sheet of $\mathfrak{R}$, that projects onto the section $[1, \infty]$ and

$$\phi = \arcsin\sqrt{\frac{\beta+1}{2}}. \tag{8.7}$$

We also have

$$\boldsymbol{u}_\alpha = \mathrm{u}_\alpha = \frac{1}{A}\int_{\mathfrak{p}_1}^{\mathfrak{p}_\alpha} \frac{\mathrm{d}x}{y} = \frac{\tau}{2} + \frac{1}{2} \tag{8.8}$$

and from (4.10) and (4.12)

$$\begin{aligned} \boldsymbol{u}_\gamma &= \mathrm{u}_\gamma = \frac{\tau}{2} + \left\{\frac{1}{2} - 2n\mathrm{u}^+\right\} \\ \boldsymbol{u}'_\gamma &= \mathrm{u}'_\gamma = \frac{\tau}{2} + \left\{\frac{1}{2} + 2n\mathrm{u}^+\right\}, \end{aligned} \tag{8.9}$$

where $\{x\}$ denotes the fractional part of the real number x.

Using (4.9) and (4.12) these results allow us to write

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \delta_n \frac{\vartheta_3^n(\mathrm{u}_x - \mathrm{u}^+ - \frac{1+\tau}{2})\vartheta_3(\mathrm{u}_x + 2n\mathrm{u}^+ - \tau)}{\vartheta_3^n(\mathrm{u}_x + \mathrm{u}^+ - \frac{1+\tau}{2})\vartheta_3(\mathrm{u}_x - \tau)} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \tilde{\delta}_n \frac{\vartheta_3^n(\mathrm{u}_x + \mathrm{u}^+ - \frac{1+\tau}{2})\vartheta_3(\mathrm{u}_x - 2n\mathrm{u}^+ - \tau)}{\vartheta_3^n(\mathrm{u}_x - \mathrm{u}^+ - \frac{1+\tau}{2})\vartheta_3(\mathrm{u}_x - \tau)}, \end{aligned}$$

with δ_n and $\tilde{\delta}_n$ constants independent of x that need to be determined. It should be noted that these expressions $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are equivalent to the genus 1 form of (4.19) and have been adopted purely because they implicitly contain a theta function expression for the integral $\Omega(\mathfrak{p}_x)$. Observing the following relations involving ϑ_1 and ϑ_3:

$$\begin{aligned} \vartheta_3\left(u - \frac{1+\tau}{2}\right) &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(u-\frac{\tau}{4})}\vartheta_1(u) \\ \vartheta_3(u-\tau) &= \mathrm{e}^{\mathrm{i}\pi(2u-\tau)}\vartheta_3(u), \end{aligned} \tag{8.10}$$

the expressions for $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ may be simplified, giving

$$\begin{aligned} \mathcal{E}_n(\mathfrak{p}_x) &= \delta_n \mathrm{e}^{\mathrm{i}2\pi n\mathrm{u}^+} \frac{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x + 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)} \\ \tilde{\mathcal{E}}_n(\mathfrak{p}_x) &= \tilde{\delta}_n \mathrm{e}^{-\mathrm{i}2\pi n\mathrm{u}^+} \frac{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x - 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)}. \end{aligned} \tag{8.11}$$

By demanding that $\mathcal{E}_n(\mathfrak{p}'_x) = \tilde{\mathcal{E}}_n(\mathfrak{p}_x)$, using the symmetry properties of the theta function, we uncover a relationship between δ_n and $\tilde{\delta}_n$, namely that

$$\tilde{\delta}_n = \delta_n \exp[\mathrm{i}4n\pi\mathrm{u}^+]. \tag{8.12}$$

An expression for δ_n follows from the fact that for $n \geqslant 1$ as $\mathfrak{p}_x \to \infty_-$, $\mathcal{E}_n(\mathfrak{p}_x) \simeq 2x^n$ and $\mathcal{E}_0 = 1$, using (5.9) we find

$$\delta_n = \begin{cases} 1 & \text{for } n = 0 \\ 2\left[\dfrac{-\vartheta_1'(0)}{A\mathrm{e}^{\mathrm{i}2\pi\mathrm{u}^+}\vartheta_1(2\mathrm{u}^+)}\right]^n \dfrac{\vartheta_3(\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} & \text{for } n \geqslant 1, \end{cases} \tag{8.13}$$

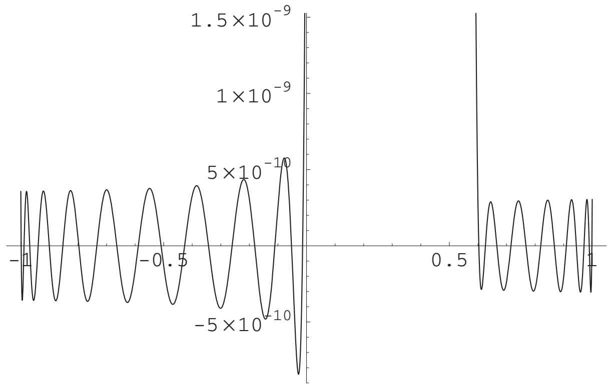

Figure 5. The genus 1 polynomial $P_{30}(x)$ for $\alpha = 0$ and $\beta = 0.6$.

from which $\tilde{\delta}_n$ also follows by (8.12).

Combining the results given above allows us to write the following expressions for the polynomials: for $n \geqslant 1$,

$$P_n(x) = \frac{1}{2}(\mathcal{E}_n(\mathfrak{p}_x) + \tilde{\mathcal{E}}_n(\mathfrak{p}_x)) = \left[-\frac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^n \frac{\vartheta_3(\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} \times \left(\frac{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x - 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)} + \frac{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x + 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)}\right) \tag{8.14}$$

and

$$Q_n(x) = \frac{w(x)}{2}(\tilde{\mathcal{E}}_n(\mathfrak{p}_x) - \mathcal{E}_n(\mathfrak{p}_x)) = \left[-\frac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^n \frac{\vartheta_3(\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} \sqrt{\frac{x-\alpha}{(x^2-1)(x-\beta)}} \times \left(\frac{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x - 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)} - \frac{\vartheta_1^n(\mathrm{u}_x - \mathrm{u}^+)\vartheta_3(\mathrm{u}_x + 2n\mathrm{u}^+)}{\vartheta_1^n(\mathrm{u}_x + \mathrm{u}^+)\vartheta_3(\mathrm{u}_x)}\right) \tag{8.15}$$

In figures 5 and 6, respectively, $P_{30}(x)$ and $Q_{30}(x)$ are plotted for the case where $\alpha = 0$ and $\beta = 0.6$. Comparing the two plots, observe that between any two zeros of $P_{30}(x)$ lies a zero of $Q_{30}(x)$, as required by theorem 1.3. Further notice that in these cases the zeros are confined to $(-1, \alpha)$ and $(\beta, 1)$.

In figure 7, we plot $P_5(x)$ and $Q_5(x)$, for $\alpha = -0.1$ and $\beta = 0.3$. Both polynomials have a single zero on the interval $\bar{E} = (-0.1, 0.3)$. Numerically evaluating $\gamma(5)$, using the expression for $\gamma(n)$ derived later in this section, we find that $\gamma(5) = 0.206\,244$ (to 6 decimal places). In accordance with theorem 7.2, it can then be clearly seen that the zero of $P_5(x)$ lies in $(-0.1, \gamma(5))$ and that of $Q_5(x)$ is contained in $(\gamma(5), 0.3)$.

Some other important quantities. We proceed by deriving explicit representations for various other important quantities required for a complete characterization of the genus 1 problem. We start with the theta function representations for the recurrence coefficients of relation (1.1). Using the fact that, as $\mathfrak{p}_x \to \infty_+$, $\mathcal{E}_n(\mathfrak{p}_x) \simeq \frac{h_n}{x^n}$, it is easily shown that

$$h_n = \delta_n \left[\frac{-\mathrm{e}^{\mathrm{i}2\pi\mathrm{u}^+}\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^n \frac{\vartheta_3((2n+1)\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)}.$$

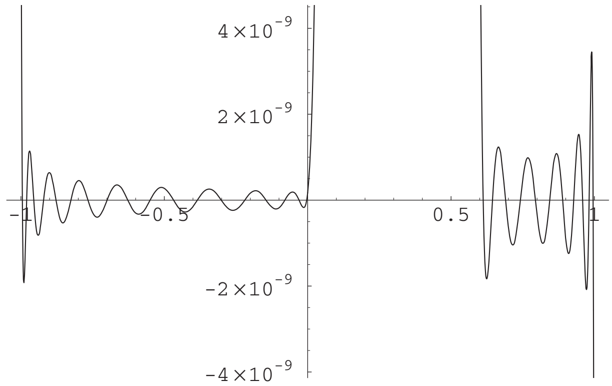

Figure 6. The genus 1 polynomial $Q_{30}(x)$ for $\alpha = 0$ and $\beta = 0.6$.

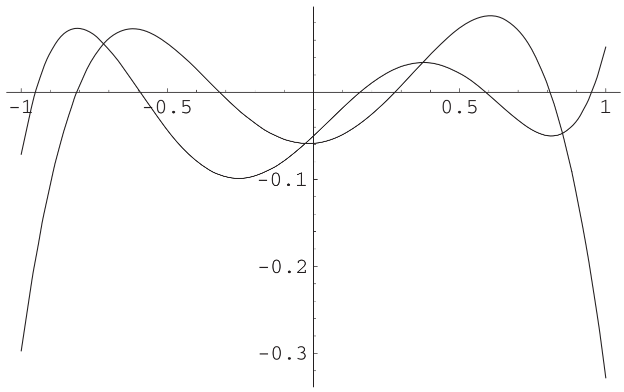

Figure 7. A plot of the polynomials, $P_5(x)$ and $Q_5(x)$ for the case where $\alpha = -0.1$ and $\beta = 0.3$, illustrating the position of their respective zeros on the interval $\bar{E} = (-0.1, 0.3)$.

From (8.13) it follows that

$$h_n = \begin{cases} 1 & \text{for } n = 0 \\ 2\left[\dfrac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^{2n} \dfrac{\vartheta_3((2n+1)\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} & \text{for } n > 0. \end{cases} \tag{8.16}$$

We recall that from the recurrence relation (1.1) and the orthogonality relation of (1.4) it can be shown that $a_n h_{n-1} = h_n$ and hence we find

$$a_n = \begin{cases} 2\left[\dfrac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^{2} \dfrac{\vartheta_3(3\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)} & \text{for } n = 1; \\ \left[\dfrac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^{2} \dfrac{\vartheta_3((2n+1)\mathrm{u}^+)\vartheta_3((2n-3)\mathrm{u}^+)}{\vartheta_3^2((2n-1)\mathrm{u}^+)} & \text{for } n \geqslant 2. \end{cases} \tag{8.17}$$

The Hankel determinant, given by $\prod_{j=1}^n h_j$, is

$$\det \mathcal{H}_n = 2^n \left[\frac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}\right]^{n(n+1)} \frac{\vartheta_3((2n+1)\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)}, \tag{8.18}$$

where

$$\mathcal{H}_n = [\mu_{j+k}]_{j,k=0,1,\ldots,n}, \tag{8.19}$$

with μ_j, the jth moment of the genus 1 weight function.

Setting $g = 1$, it then follows from (5.17), using the results of (4.21), (8.8) and (8.10) that

$$p_1(n) = \frac{1}{A}\frac{\vartheta_3'((2n-1)\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} - \frac{(2n-1)}{A}\frac{\vartheta_3'(\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)} - \frac{n(\beta-\alpha)}{2}. \tag{8.20}$$

It is worth noting at this stage that we have now shown that

$$P_1(x) = x + \frac{\alpha-\beta}{2}.$$

When $\alpha > 0$ selecting $\beta > 3\alpha$ and for $\alpha < 0$ choosing $\beta > -\alpha$, we ensure that

$$\alpha < \frac{\beta-\alpha}{2} < \beta.$$

Hence we have demonstrated analytically the existence of cases where the polynomials have zeros on the interval $\bar{E}$. We can now determine an expression for the other recurrence coefficient b_n. Again we use the fact that $b_n = p_1(n-1) - p_1(n)$:

$$b_n = \frac{\beta-\alpha}{2} + \frac{1}{A}\left[2\frac{\vartheta_3'(\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)} + \frac{\vartheta_3'((2n-3)\mathrm{u}^+)}{\vartheta_3((2n-3)\mathrm{u}^+)} - \frac{\vartheta_3'((2n-1)\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)}\right], \qquad n \geqslant 1. \tag{8.21}$$

We proceed with the determination of $\gamma(n)$. Previously in deriving an expression for the c_j, $j = 0, \ldots, g$, we noticed that, as $\mathfrak{p}_x \to \infty_-$,

$$\mathcal{E}_n(\mathfrak{p}_x) = 2P_n(x) + \mathrm{O}(x^{-n}). \tag{8.22}$$

We had shown from (6.26) that

$$y\mathcal{E}_n'(x) - \mathcal{E}_n(\mathfrak{p}_x)\left[\sum_{j=0}^{g} c_j x^j + \frac{y}{2}\sum_{j=1}^{g}\left(\frac{1}{x-\gamma_j} - \frac{1}{z-\alpha_j}\right) + \frac{1}{2}\sum_{j=1}^{g}\frac{y_j}{x-\gamma_j}\right] = 0, \tag{8.23}$$

for all x. Expanding the left-hand side around the point at ∞_-, using (8.22) and setting the first g coefficients equal to zero, we then obtained relationships for the c_j in terms of $\{p_k, \gamma_k, \alpha_k, \beta_k : k = 1, \ldots, g\}$. In the particular case where $g = 1$, we found that, for $n > 0$, $c_1(n) = -n$ and

$$c_0(n) = p_1(n) + n\frac{\alpha+\beta}{2} + \frac{\gamma(n)-\alpha}{2}. \tag{8.24}$$

We shall now use the same principle but this time we expand around the point at ∞_+. In so doing we obtain a second relationship between $c_0(n)$ and $\gamma(n)$. Eliminating $c_0(n)$ between this and the equation above will provide the expression for $\gamma(n)$ that we seek. Firstly, we require the asymptotic expansion of $\mathcal{E}_n(\mathfrak{p}_x)$ as $\mathfrak{p}_x \to \infty_+$. From (1.13) it follows that in this limit

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{1}{w(x)}\left[\frac{1}{x^{n+1}}\int_E P_n(t)t^n p(t)\,\mathrm{d}t + \frac{1}{x^{n+2}}\int_E P_n(t)t^{n+1}p(t)\,\mathrm{d}t + \mathrm{O}(x^{-n-3})\right]. \tag{8.25}$$

Observe that we can always write

$$t^n = \sum_{j=0}^{n}\lambda_j(n)P_{n-j}(t), \tag{8.26}$$

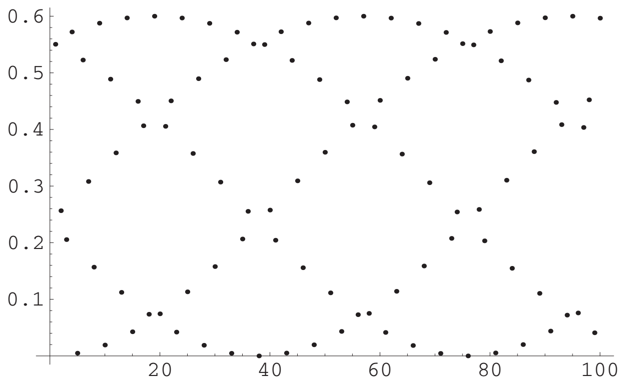

Figure 8. The values of $\gamma(n)$ as n varies from 1 to 100, when $\alpha = 0$ and $\beta = 0.6$.

where it is easily shown that $\lambda_0(n) = 1$ and $\lambda_1(n) = -p_1(n)$. It then follows from the orthogonality relationship that

$$\int_E P_n(t)t^n p(t)\,\mathrm{d}t = h_n$$

$$\int_E P_n(t)t^{n+1} p(t)\,\mathrm{d}t = \lambda_1(n+1)h_n.$$

In the case where $g = 1$, as $x \to \infty_+$ we have

$$\frac{1}{w(x)} = x + \frac{\alpha - \beta}{2} + \mathrm{O}(x^{-1}),$$

so that ultimately in this limit we obtain

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{h_n}{x^n} + \frac{h_n}{x^{n+1}}\left(\frac{\alpha - \beta}{2} + \lambda_1(n+1)\right) + \mathrm{O}(x^{-n-2}). \tag{8.27}$$

We then expand the left-hand member of (8.23), with $g = 1$, around the point at ∞_+ and set the coefficients in this expansion to zero. The coefficient of x^{-n+1} verifies the fact that $c_1(n) = -n$, while that of x^{-n} gives the relation

$$c_0(n) = n\frac{\alpha + \beta}{2} + \frac{\beta - \gamma(n)}{2} - \lambda_1(n+1). \tag{8.28}$$

Subtracting (8.24) from this expression we find that

$$\gamma(n) = p_1(n+1) - p_1(n) + \frac{\beta + \alpha}{2}.$$

Then, since $b_n = p_1(n-1) - p_1(n)$, it follows that

$$\gamma(n) = \frac{\beta + \alpha}{2} - b_{n+1} = \alpha + \frac{1}{A}\left[\frac{\vartheta_3'((2n+1)\mathrm{u}^+)}{\vartheta_3((2n+1)\mathrm{u}^+)} - \frac{\vartheta_3'((2n-1)\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)} - 2\frac{\vartheta_3'(\mathrm{u}^+)}{\vartheta_3(\mathrm{u}^+)}\right], \tag{8.29}$$

by virtue of (8.21). In figures 8 and 9 we plot the values that $\gamma(n)$ takes as n varies for two specific choices of the interval E. Note that in both cases $\gamma(n) \in [\alpha, \beta]$ for all values of n, as required by theorem 3.1.

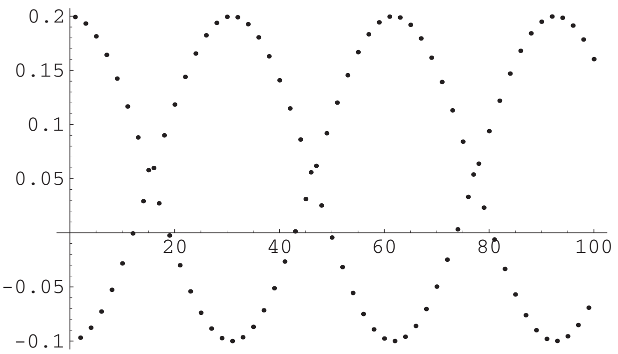

Figure 9. The values of $\gamma(n)$ as n varies from 1 to 100, when $\alpha = -0.1$ and $\beta = 0.2$.

A convenient expression for $c_0(n)$. We take this opportunity to determine another expression for $c_0(n)$ that will prove of great use later. Previously we have obtained representations by essentially expanding different expressions for $(\mathrm{d}/\mathrm{d}x)\ln\mathcal{E}_n$ around the point at ∞_- and ∞_+ and then equating. Again we adopt a similar approach, this time expanding for $\mathfrak{p}_x$ around 1 in terms of the local coordinate $\xi = \sqrt{x-1}$. Note that in this locality it follows straightforwardly from the definition for u_x that

$$\mathrm{u}_x = \frac{\sqrt{2}}{A\sqrt{(1-\beta)(1-\alpha)}}\xi + \mathrm{O}(\xi^3). \tag{8.30}$$

From the expression for $\mathcal{E}_n$ given in (8.11) we have

$$\ln\mathcal{E}_n(\mathfrak{p}_x) = \ln(\delta_n \mathrm{e}^{\mathrm{i}2\pi n\mathrm{u}^+}) + n[\ln\vartheta_1(\mathrm{u}_x - \mathrm{u}^+) - \ln\vartheta_1(\mathrm{u}_x + \mathrm{u}^+)] \\ + \ln\vartheta_3(\mathrm{u}_x + 2n\mathrm{u}^+) - \ln\vartheta_3(\mathrm{u}_x).$$

Hence by using (5.9) and (8.30) it is easily shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n(\mathfrak{p}_x) = \frac{\frac{\vartheta_3'(2n\mathrm{u}^+)}{\vartheta_3(2n\mathrm{u}^+)} - 2n\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)}}{A\sqrt{2(1-\beta)(1-\alpha)}}\frac{1}{\xi} + \mathrm{O}(1). \tag{8.31}$$

On the other hand, expanding the $g = 1$ form of (6.26) in terms of ξ, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\mathcal{E}_n(\mathfrak{p}_x) = \frac{c_0 - n + \frac{y_1}{2(1-\gamma)}}{\sqrt{2(1-\beta)(1-\alpha)}}\frac{1}{\xi} + \mathrm{O}(1). \tag{8.32}$$

By equating the last two local expressions we determine the following representation for $c_0(n)$:

$$c_0 = n - \frac{y_1}{2(1-\gamma)} + \frac{1}{A}\left(\frac{\vartheta_3'(2n\mathrm{u}^+)}{\vartheta_3(2n\mathrm{u}^+)} - 2n\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)}\right). \tag{8.33}$$

It is easily shown using results in [28, ch 9] that $\vartheta_3'(u)/\vartheta_3(u)$ is a bounded function for all real values of u and that $\vartheta_1'(u)/\vartheta_1(u)$ has its only real valued singularities at $u \in \mathbb{Z}$. Thus, since $\gamma \in [\alpha, \beta]$ for all values of n and $\mathrm{u}^+ \notin \mathbb{Z}$, we have shown that $c_0(n)$ has the general form $c_0(n) = n\kappa_1 + \kappa_2(n)$, where κ_1 is finite and constant for a fixed n and $\kappa_2(n)$ is bounded for all values of n. Notice in particular that

$$\lim_{n\to\infty}\frac{c_0(n)}{n} = 1 - \frac{2}{A}\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)}. \tag{8.34}$$

Contraction to one interval. We continue by demonstrating that allowing $\alpha \to \beta$, so that E becomes a single interval, leads to the recovery of well known results for the Chebyshev polynomials. Firstly let us examine the differential equation satisfied by $P_n(x)$ and $R_n(x)$:

$$Y''(x) + s(x)Y'(x) + t(x)Y(x) = 0,$$

where

$$\begin{aligned} s(x) &:= -\left(2f_1 + \frac{f_2'}{f_2} + \frac{w'}{w}\right) \\ t(x) &:= f_1^2 - f_1' + f_1\left(\frac{f_2'}{f_2} + \frac{w'}{w}\right) - f_2^2 w^2, \end{aligned} \tag{8.35}$$

with

$$\begin{aligned} f_1(x) &= \frac{1}{2}\left(\frac{1}{x-\gamma} - \frac{1}{x-\alpha}\right) \\ f_2(x) &= \frac{1}{x-\alpha}\left(nx - c_0 - \frac{y_1}{2(x-\gamma)}\right) \\ w(x) &= \sqrt{\frac{x-\alpha}{(x^2-1)(x-\beta)}}. \end{aligned}$$

Note that in the case where we set $\alpha = \beta = \gamma(n)$, thus reducing the problem to a single interval, since the weight function is then symmetrical on $[-1, 1]$, it follows from the orthogonality relationship (1.4) that $p_1(n) = 0$ for all values of n. Consequently, using (8.24), we find that $c_0(n) = n\alpha$, $n \geqslant 0$. Thus $f_1 = 0$, $f_2 = n$ and $w = 1/\sqrt{x^2-1}$ and we recover the classical equation satisfied by the Chebyshev polynomials of the first kind, $T_n(x)$:

$$(x^2-1)Y''(x) + xY_n'(x) - n^2Y(x) = 0.$$

In order to determine the nature of $P_n(x)$ and $Q_n(x)$ as $\alpha \to \beta$, we must first understand the behaviour of the fundamental quantities A, τ and u^+ in this limit. Setting $\beta = \alpha + \delta$ and allowing $\delta \to 0$, it follows from (8.2) that

$$A = \frac{2\pi}{\sqrt{1-\alpha^2}} + \mathrm{O}(\delta), \tag{8.36}$$

and (8.4) leads to

$$\tau = -\frac{\mathrm{i}}{\pi}\ln\delta + \frac{\mathrm{i}}{\pi}\ln[8(1-\alpha^2)] + \mathrm{O}(\delta). \tag{8.37}$$

In the limit $\delta \to 0$, we also have

$$\mathrm{u}^+ = \frac{1}{\pi}\arcsin\sqrt{\frac{\alpha+1}{2}} + \mathrm{O}(\delta). \tag{8.38}$$

From the definition for $\vartheta_1(u)$ given in (8.1) it follows that

$$\begin{aligned} \vartheta_1'(0) &= \frac{2\pi}{[8(1-\alpha^2)]^{1/4}}\delta^{1/4} + \mathrm{O}(\delta^{5/4}) \\ \vartheta_1(2\mathrm{u}^+) &= \frac{2\sin 2\pi\mathrm{u}^+}{[8(1-\alpha^2)]^{1/4}}\delta^{1/4} + \mathrm{O}(\delta^{5/4}). \end{aligned}$$

Thus recalling (8.14), we have determined that

$$\varrho(n) = (-1)^n 2^{-n} + \mathrm{O}(\delta). \tag{8.39}$$

When $\beta = \alpha + \delta$,

$$\mathrm{u}_x = \frac{1}{A}\left(\int_1^x \frac{\mathrm{d}x}{(x-\alpha)\sqrt{(x^2-1)}} + \mathrm{O}(\delta)\right).$$

integration then gives

$$\mathrm{u}_x = \frac{1}{\pi}\arcsin\sqrt{\frac{(\alpha+1)(x-1)}{2(x-\alpha)}} + \mathrm{O}(\delta). \tag{8.40}$$

From the theta function definitions of (8.1) it follows from (8.14) that

$$g_1(x;n) = \left[\frac{\sin\pi(\mathrm{u}_x + \mathrm{u}^+)}{\sin\pi(\mathrm{u}_x - \mathrm{u}^+)}\right]^n + \mathrm{O}(\delta) \tag{8.41}$$

and

$$g_2(x;n) = \left[\frac{\sin\pi(\mathrm{u}_x - \mathrm{u}^+)}{\sin\pi(\mathrm{u}_x + \mathrm{u}^+)}\right]^n + \mathrm{O}(\delta). \tag{8.42}$$

Using (8.38) and (8.40), after some elementary trigonometric manipulations,

$$g_1(x;n) = (-1)^n\left(\frac{\sqrt{1+x} - \mathrm{i}\sqrt{1-x}}{\sqrt{1+x} + \mathrm{i}\sqrt{1-x}}\right)^n + \mathrm{O}(\delta) = (-1)^n(x - \mathrm{i}\sqrt{1-x^2})^n + \mathrm{O}(\delta) \tag{8.43}$$

and hence

$$g_2(x;n) = (-1)^n\left(x - \mathrm{i}\sqrt{1-x^2}\right)^{-n} + \mathrm{O}(\delta). \tag{8.44}$$

Setting $x = \cos z$ implies that

$$g_1(\cos z; n) + g_2(\cos z; n) = (-1)^n[\mathrm{e}^{-\mathrm{i}nz} + \mathrm{e}^{\mathrm{i}nz}] + \mathrm{O}(\delta),$$

and thus

$$\lim_{\alpha\to\beta} P_n(\cos z) = 2^{1-n}\cos nz; \qquad n \geqslant 1. \tag{8.45}$$

Similarly we find that

$$\lim_{\alpha\to\beta} Q_n(\cos z) = 2^{1-n}\frac{\sin nz}{\sin z}. \tag{8.46}$$

Both of these expressions are in agreement with the results for the Chebyshev polynomials of the first and second kinds, respectively [23].

9. Determination of the $\{\gamma_i\}$ in higher genus cases

The importance of the points $\{\mathfrak{p}_{\gamma_j} : j = 1, \ldots, g\}$ is apparent from the preceding sections. Originating from the consideration of the divisor for Akhiezer's function

$$\mathcal{E}_n(\mathfrak{p}_x) = P_n(x) - \frac{Q_n(x)}{w(x)}$$

on the Riemann surface $\mathfrak{R}$, they are required in order to completely quantify the differential equations satisfied by the polynomials P_n and Q_n, and indeed any quantity that has a dependence on y_j, $j = 1, \ldots, g$. The $\mathfrak{p}_{\gamma_j}$ represent the solution to the Jacobi inversion problem of equation (4.10) and as such are the zeros of the Riemann theta function of g variables:

$$\vartheta\left(\boldsymbol{u}_x - \boldsymbol{C} - \sum_{j=1}^{g}\boldsymbol{u}_{\alpha_j} - n(\boldsymbol{u}^- - \boldsymbol{u}^+)\right).$$

In general this result is not easy to work with.

In an effort to advance our understanding of these points, in this section we explicitly consider the determination of $\{\gamma_j : j = 1, \ldots, g\}$, their projection on the complex plane, for the case where $g = 2$. By taking different representations of the asymptotic expansion of $\frac{\mathrm{d}}{\mathrm{d}x} \ln \mathcal{E}_n(\mathfrak{p}_x)$ around the points at ∞_- and ∞_+, matching coefficients we generalize the technique used in the genus 1 case and obtain a quadratic equation satisfied by γ_1 and γ_2. As the reader will see this technique can, in principle, be extended, so that in the general genus g case, a polynomial of degree g satisfied by the $\{\gamma_j : j = 1, \ldots, g\}$ is found.

The case where we expand around the point at ∞_- has been considered previously, when we derived expressions for the c_j in terms of $\{\gamma_k, \alpha_k, \beta_k : k = j, \ldots, g\}$ in section 6. When $g = 2$, we find the following expressions:

$$\begin{aligned} c_2 &= -n \\ c_1 &= p_1(n) - n\mathsf{b}_1 + \frac{\gamma_1(n) - \alpha_1 + \gamma_2(n) - \alpha_2}{2} \\ c_0 - \mathsf{b}_1 c_1 &= 2p_2(n) - p_1^2(n) + n(\mathsf{b}_1^2 - \mathsf{b}_2) + \frac{\gamma_1^2(n) - \alpha_1^2 + \gamma_2^2(n) - \alpha_2^2}{2}, \end{aligned} \tag{9.1}$$

where the b_j are defined as in (6.22). As $g = 2$ in the case under consideration we may write, as $\mathfrak{p}_x \to \infty_-$, that

$$y = -x^3 - \mathsf{b}_1 x^2 - \mathsf{b}_2 x + \mathrm{O}(1), \tag{9.2}$$

with

$$\begin{aligned} \mathsf{b}_1 &= -\frac{\alpha_1 + \beta_1 + \alpha_2 + \beta_2}{2} \\ \mathsf{b}_2 &= \frac{2[\alpha_1(\alpha_2 + \beta_1 + \beta_2) + \alpha_2(\beta_1 + \beta_2) + \beta_1\beta_2] - (\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2)}{8} - \frac{1}{2}. \end{aligned} \tag{9.3}$$

Considering the expansion of $\mathcal{E}_n(\mathfrak{p}_x)$ as $\mathfrak{p}_x \to \infty_+$, from (1.13) we find that

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{1}{w(x)}\left[\frac{\int_E P_n(t)t^n p(t)\,\mathrm{d}t}{x^{n+1}} + \frac{\int_E P_n(t)t^{n+1} p(t)\,\mathrm{d}t}{x^{n+2}} + \frac{\int_E P_n(t)t^{n+2} p(t)\,\mathrm{d}t}{x^{n+3}} + \mathrm{O}(x^{-n-4})\right].$$

Writing, as we did in section 8,

$$t^n = \sum_{j=0}^{n} \lambda_j(n) P_{n-j}(t), \tag{9.4}$$

it is easily shown that $\lambda_0(n) = 1$, $\lambda_1(n) = -p_1(n)$ and $\lambda_2(n) = p_1(n)p_1(n-1) - p_2(n)$. It then follows from the orthogonality relationship satisfied by $P_n(x)$ that

$$\mathcal{E}_n(\mathfrak{p}_x) = \frac{h_n}{w(x)}\left[\frac{1}{x^{n+1}} + \frac{\lambda_1(n+1)}{x^{n+2}} + \frac{\lambda_2(n+2)}{x^{n+3}} + \mathrm{O}(x^{-n-4})\right]. \tag{9.5}$$

We proceed by observing that in this case

$$\ln w(x) = \tfrac{1}{2}[\ln(x - \alpha_1) + \ln(x - \alpha_2) - \ln(x - \beta_1) - \ln(x - \beta_2) - \ln((x^2 - 1)].$$

It then follows that, as $\mathfrak{p}_x \to \infty_+$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{1}{w(x)} = \frac{1}{x} + \frac{\beta_1 - \alpha_1 + \beta_2 - \alpha_2}{2x^2} + \frac{1}{x^3}\left(1 + \frac{\beta_1^2 - \alpha_1^2 + \beta_2^2 - \alpha_2^2}{2}\right) + \mathrm{O}(x^{-4}).$$

Combining this result with that of (9.5), we find the following asymptotic expansion:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln \mathcal{E}_n(\mathsf{p}_x) = -\frac{n}{x} + \frac{1}{x^2}\left(\frac{\beta_1-\alpha_1+\beta_2-\alpha_2}{2} - \lambda_1(n+1)\right) + \frac{1}{x^3}\left(1+\frac{\beta_1^2-\alpha_1^2+\beta_2^2-\alpha_2^2}{2} + \lambda_1^2(n+1) - 2\lambda_2(n+2)\right) + \mathrm{O}(x^{-4}). \tag{9.6}$$

Recall from (6.26) that for all values of x, in the case where $g = 2$,

$$y\frac{\mathrm{d}}{\mathrm{d}x}\ln \mathcal{E}_n(\mathsf{p}_x) = c_0 + c_1x + c_2x^2 + \frac{y}{2}\sum_{i=1}^{2}\left(\frac{1}{x-\gamma_i} - \frac{1}{x-\alpha_i}\right) + \frac{1}{2}\sum_{i=1}^{2}\frac{y_i}{x-\gamma_i}.$$

Expanding each side of this identity around the point at ∞_+ using (9.6), and equating the coefficients, provides the following expressions:

$$\begin{aligned} c_1 &= -\lambda_1(n+1) - n\mathsf{b}_1 + \frac{\beta_1-\gamma_1(n)+\beta_2-\gamma_2(n)}{2} \\ c_0 - \mathsf{b}_1c_1 &= \lambda_1^2(n+1) - 2\lambda_2(n+2) + n(\mathsf{b}_1^2-\mathsf{b}_2) + 1 + \frac{\beta_1^2-\gamma_1^2(n)+\beta_2^2-\gamma_2^2(n)}{2}. \end{aligned} \tag{9.7}$$

From equations (9.1) and (9.7) it follows that

$$\begin{aligned} \gamma_1(n)+\gamma_2(n) &= F(n) \\ \gamma_1^2(n)+\gamma_2^2(n) &= G(n) \end{aligned} \tag{9.8}$$

where

$$\begin{aligned} F(n) &:= p_1(n+1) - p_1(n) + \frac{\alpha_1+\beta_1+\alpha_2+\beta_2}{2} \\ G(n) &:= 1 + p_1^2(n+1) + p_1^2(n) - 2[p_1(n+2)p_1(n+1) - p_2(n+2) + p_2(n)] \\ &\quad + \frac{\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2}{2}, \end{aligned} \tag{9.9}$$

having used the previous expressions for λ_1 and λ_2 in terms of p_1 and p_2. It is then easily shown that $\gamma_1(n)$ and $\gamma_2(n)$ are the solutions of

$$x^2 - F(n)x + \frac{F^2(n)-G(n)}{2} = 0. \tag{9.10}$$

Since the p_j can be expressed in terms of the recurrence coefficients of relation (1.1), we have thus demonstrated how to find expressions for γ_1 and γ_2 in terms of the $\{a_j\}$ and $\{b_j\}$.

10. Large n asymptotics

An object which will play an important role in the following development is $k_n(x,x)$, the Christoffel–Darboux kernel evaluated at the same point. For our purposes, we introduce

$$\sigma_n(x) := p(x)k_n(x,x) = \frac{p(x)}{h_{n-1}}(P_n'(x)P_{n-1}(x) - P_n(x)P_{n-1}'(x)). \tag{10.1}$$

In the case of the Chebyshev polynomials, where $p(x) = 1/\sqrt{1-x^2}$, $T_n(\cos\theta) = 2^{1-n}\cos n\theta$ and $h_n = \pi 2^{1-2n}$,

$$\sigma_n(x) = \frac{1}{\pi(1-x^2)}[n\cos[(n-1)\arccos x]\sin[n\arccos x] - (n-1)\cos[n\arccos x]\sin[(n-1)\arccos x]].$$

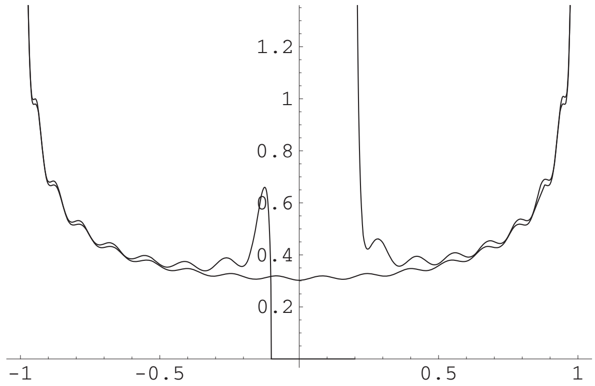

Figure 10. $\sigma_n(x)/n$ for the genus 1 polynomial P_n, where $\alpha = -0.1$ and $\beta = 0.2$, and T_n, the Chebyshev polynomial of the first kind. In both cases $n = 20$.

Observe that this may be written as

$$\sigma_n(x) = \frac{n\sqrt{1-x^2} + \sin[(n-1)\arccos x]\cos[n\arccos x]}{\pi(1-x^2)}. \tag{10.2}$$

from which it follows that

$$\lim_{n\to\infty} \frac{\sigma_n(x)}{n} = \frac{1}{\pi\sqrt{1-x^2}}, \qquad -1 < x < 1. \tag{10.3}$$

Observe that in the context of the material that follows, the right-hand side of (10.3) can be identified as the equilibrium density of the interval $[-1, 1]$.

In order to determine a corresponding expression for the generalized Chebyshev polynomials, we recall that (6.32) states

$$P_n'(x) = f_1(x; n)P_n(x) + f_2(x; n)Q_n(x),$$

with

$$\begin{aligned} f_1(x; n) &:= \frac{1}{2}\sum_{j=1}^{g}\left(\frac{1}{x-\gamma_j(n)} - \frac{1}{x-\alpha_j}\right) \\ f_2(x; n) &:= \frac{nx^g - \sum_{j=0}^{g-1} c_j(n)x^j - \frac{1}{2}\sum_{j=1}^{g}\frac{y_j}{x-\gamma_j(n)}}{\prod_{j=1}^{g}(x-\alpha_j)}. \end{aligned} \tag{10.4}$$

We can use this to write

$$\begin{aligned} \sigma_n(x) = \frac{p(x)}{h_{n-1}}\bigg[& P_n(x)P_{n-1}(x)(f_1(x; n) - f_1(x; n-1)) \\ & + f_2(x; n)Q_n(x)P_{n-1}(x) - f_2(x; n-1)P_n(x)Q_{n-1}\bigg]. \end{aligned} \tag{10.5}$$

σ_n/n for P_n in the genus 1 case, where $\alpha = -0.1$, $\beta = 0.2$ and $n = 20$ is illustrated in figure 10. It is contrasted with $\sigma_n(x)/n$ of the Chebyshev polynomials of the first kind, also plotted for $n = 20$. Observe the similarity near the edges (i.e. near to ± 1) and the contrast between the behaviour at α and the other end points for the generalized polynomial, which is dictated by the weight function factor.

An electrostatic problem. Let σ be a positive charge density defined on the set $E = [-1, \alpha_1] \cup_{j=1}^{g-1} [\beta_j, \alpha_j] \cup [\beta_g, 1]$, satisfying the condition that

$$\int_E \sigma(x)\,\mathrm{d}x = 1. \tag{10.6}$$

We consider the energy functional

$$I[\sigma] = -\int_E \int_E \ln|x - t|\sigma(x)\sigma(t)\,\mathrm{d}x\,\mathrm{d}t. \tag{10.7}$$

The equilibrium density of the set E [27, p55], denoted by σ^*, is obtained by minimizing the energy functional subject to the normalization constraint. Thus, by taking the functional derivative of (10.7), σ^* satisfies the following integral equation:

$$V = -P\int_E \ln|x - t|\sigma^*(t)\,\mathrm{d}t. \tag{10.8}$$

The Lagrange multiplier V is known as the conductor potential of the set E, and is a constant for $x \in E$. Differentiating (10.8) with respect to x, σ^* then satisfies

$$P\int_E \frac{\sigma^*(z)}{x - t}\,\mathrm{d}t = 0, \qquad x \in E. \tag{10.9}$$

The general solution to this equation is then found to be [11]

$$\sigma^*(x) = \frac{U_g(x)}{y}, \qquad x \in E, \tag{10.10}$$

with $y = \sqrt{(x^2 - 1)\prod_{j=1}^{g}(x - \alpha_i)(x - \beta_i)}$ and U_g an arbitrary polynomial of degree g. Note that without loss of generality we assume that y takes values that are consistent with approaching E from above on the +sheet of $\mathfrak{R}$. The requirement that $\sigma^*(x)$ be real and positive for all $x \in E$ demands that the zeros of U_g lie one a piece in each of the intervals that make up $\bar{E}(:= [-1, 1]\backslash E)$. Quantitatively, the $g + 1$ coefficients of U_g are determined by the fact that $\int_E \sigma^*(x)\mathrm{d}x = 1$ and the Akhiezer–Widom condition [3, 30, section 13], which states

$$\int_{\alpha_i}^{\beta_i} \sigma^*(x)\,\mathrm{d}x = 0, \qquad i = 1, \ldots, g, \tag{10.11}$$

where, for $x \in \bar{E}$, $\sigma^*(x)$ is taken to be the extension of the function given in (10.10). These conditions are required to ensure that the conductor potential V is the same constant on each sub-interval and can easily be demonstrated as follows. We introduce the function [16, 30, section 13]

$$\Upsilon(x) = \mathrm{Re}\left(-\int_E \sigma^*(t)\ln(x - t)\,\mathrm{d}t\right), \qquad x \in \mathbb{C}.$$

When $x \in E$ we see that this function is, in fact, a constant equal to V, so that $(\mathrm{d}/\mathrm{d}x)\Upsilon(x) = 0$. However, when $x \in \bar{E}$ we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Upsilon(x) = \int_E \frac{\sigma^*(t)}{t - x}\,\mathrm{d}t = \mathrm{i}\pi\sigma^*(x),$$

where the integral above is evaluated in exactly the same way as $w(x)$ was determined in (2.2). In light of this last fact we find that, in order for V to be the same constant on each of the intervals making up E, we require that $\Upsilon(\beta_i) - \Upsilon(\alpha_i) = 0$, $i = 1, \ldots, g$, and consequently this implies the conditions of (10.11).

Notice that from the normalization condition it is easily shown that

$$U_g(x) = \frac{\mathrm{i}}{\pi}\left(x^g + \sum_{j=0}^{g-1} k_j x^j\right), \tag{10.12}$$

where the coefficients, k_j, follow from (10.11). Hence, referring to (4.14) and (4.16) we have identified that the Akhiezer–Widom conditions are equivalent to the conventional normalization of Ω. Indeed we have shown that

$$\sigma^*(x) = \frac{\mathrm{i}}{\pi}\frac{\mathrm{d}\Omega}{\mathrm{d}x}. \tag{10.13}$$

Returning to consider the $g = 1$ case explicitly, the equilibrium density is given by

$$\sigma^*(x) = \frac{\mathrm{i}(x + k_0)}{\pi\sqrt{(x^2-1)(x-\alpha)(x-\beta)}}, \qquad x \in E. \tag{10.14}$$

where

$$k_0 = -\frac{\int_\alpha^\beta \frac{x\,\mathrm{d}x}{y}}{\int_\alpha^\beta \frac{\mathrm{d}x}{y}}. \tag{10.15}$$

These integrals can be written in terms of the standard elliptic integral functions [12]

$$k_0 := \frac{(1-\beta)\Pi\left(\frac{\beta-\alpha}{1-\alpha}, k\right)}{K(k)} - 1, \tag{10.16}$$

with $\Pi(m, k)$ a complete elliptic integral of the third kind and k defined as in (8.3). We now take the opportunity to recast k_0 into a form that will prove convenient later. In order to make comparisons with the preceding results we will require a representation in terms of theta functions.

In cases where $0 < m < k^2$ the elliptic integral of the third kind,

$$\Pi(\psi, m, k) := \int_0^{\sin\psi} \frac{\mathrm{d}t}{(1-mt^2)\sqrt{(1-t^2)(1-k^2t^2)}},$$

can be written as [28, ch 10]

$$\Pi(\psi, m, k) = -\frac{\mathrm{sn}r}{\mathrm{cn}r\,\mathrm{dn}r}\left[\frac{1}{2}\ln\frac{\vartheta_4\left(\frac{u+r}{2K}\right)}{\vartheta_4\left(\frac{u-r}{2K}\right)} - u\frac{\mathrm{d}}{\mathrm{d}r}\ln\vartheta_1\left(\frac{r}{2K}\right)\right], \tag{10.17}$$

where sn, cn and dn are the Jacobian elliptic functions defined by the relations

$$u = \int_0^{\mathrm{sn}u} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{10.18}$$

and

$$\begin{aligned} &\mathrm{cn}^2 u + \mathrm{sn}^2 u = 1 \\ &\mathrm{dn}^2 u + k^2\mathrm{sn}^2 u = 1, \end{aligned} \tag{10.19}$$

where u and r are given by

$$\begin{aligned} &m = k^2\mathrm{sn}^2 r \\ &\sin\psi = \mathrm{sn}u. \end{aligned} \tag{10.20}$$

Note that

$$0 < \frac{\beta-\alpha}{1-\alpha} < k^2.$$

Thus, by observing that $\mathrm{sn}^{-1}x = F(\arcsin x, k)$, we may write $u = \mathrm{sn}^{-1}1 = K$ and

$$r = F\left(\arcsin \frac{1}{k}\sqrt{\frac{\beta-\alpha}{1-\alpha}}, k\right) = F(\phi, k),$$

with ϕ as defined in (8.7). Consequently we find that

$$\Pi\left(\frac{\beta-\alpha}{1-\alpha}, k\right) = \frac{\sqrt{(1-\alpha)(1+\beta)}}{2(1-\beta)} \frac{\vartheta_1'\left(\frac{r}{2K}\right)}{\vartheta_1\left(\frac{r}{2K}\right)},$$

since $\vartheta_4\left(u+\frac{1}{2}\right) = \vartheta_3(u)$. From (8.6), $\frac{r}{2K} = \mathrm{u}^+$ and so that

$$k_0 = \frac{2}{A}\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)} - 1 \tag{10.21}$$

and hence observe, with reference to (8.34), that

$$k_0 = -\lim_{n\to\infty} \frac{c_0(n)}{n}. \tag{10.22}$$

It will become apparent that V is a quantity of great importance. We now proceed by simplifying the expression for the potential in (10.8), which by using (10.10) is given by

$$V = -\int_E \frac{U_g(x)}{y} \ln(1-x)\,\mathrm{d}x. \tag{10.23}$$

Using the convenient integral representation

$$\ln(1-x) = \int_0^1 \frac{x\,\mathrm{d}\lambda}{x\lambda - 1},$$

it follows that

$$V = -\frac{\mathrm{i}}{\pi}\int_0^1 \frac{\mathrm{d}\lambda}{\lambda}\int_E \frac{x(x^g + k_{g-1}x^{g-1} + \cdots + k_0)}{\left(x-\frac{1}{\lambda}\right)y}\,\mathrm{d}x.$$

The second integral on the right-hand side is easily evaluated; using the contour Λ as defined in section 2 and Cauchy's integral theorems

$$\int_E \frac{x^j}{y}\,\mathrm{d}x = \begin{cases} 0 & \text{for } j < g \\ -\mathrm{i}\pi & \text{for } j = g, \end{cases}$$

and for $\frac{1}{\lambda} \notin E$

$$\int_E \frac{x^g + k_{g-1}x^{g-1} + \cdots + k_0}{\left(x-\frac{1}{\lambda}\right)y}\,\mathrm{d}x = \mathrm{i}\pi \frac{\lambda^{-g} + k_{g-1}\lambda^{1-g} + \cdots + k_0}{\sqrt{\left(\frac{1}{\lambda^2}-1\right)\prod_{j=1}^g\left(\frac{1}{\lambda}-\alpha_j\right)\left(\frac{1}{\lambda}-\beta_j\right)}}.$$

If $t = \frac{1}{\lambda}$, then

$$V = \int_1^\infty \left(\frac{t^g + k_{g-1}t^{g-1} + \cdots + k_0}{\sqrt{(t^2-1)\prod_{j=1}^g (t-\alpha_j)(t-\beta_j)}} - \frac{1}{t}\right)\mathrm{d}t. \tag{10.24}$$

Recalling (5.6) it is evident that $V = \chi_0$.

If S is a closed bounded set in the complex plane that contains infinitely many points, then taking n of these points $\{x_j : j = 1, \ldots, n\}$ and writing

$$d_n := \left(\max_{x_j\in S}\left\{\prod_{j<k}^{1,\ldots,n} |x_j - x_k|\right\}\right)^{1/(n/2)}, \tag{10.25}$$

Fekete [9] showed the limit as $n \to \infty$ of the sequence $\{d_n\}$ exists and called this quantity the transfinite diameter of the set S, denoted $C(S)$. The proof of this existence theorem is also presented in [27, theorem 21, p 71]. Considering the set of monic polynomials of degree n, denoted by $\{\pi_n(x)\}$, and taking

$$M_n := \min_{\pi_n}\{\max_{x\in S} |\pi_n(x)|\}, \tag{10.26}$$

the transfinite diameter of S can then be identified [9] as the limit

$$C(S) = \lim_{n\to\infty} M_n^{1/n}. \tag{10.27}$$

It is shown in [27, theorem 23, p 72] that there exists a unique polynomial $t_n(x) \in \{\pi_n(x)\}$ with the property that

$$\max_{x\in E} |t_n(x)| = M_n. \tag{10.28}$$

In the literature, t_n is often referred to as the Chebyshev polynomial of degree n for the set E. However, in this paper we reserve this title for the classical polynomials orthogonal on the interval $[-1, 1]$ and simply refer to t_n as the extremal polynomial associated with the set E.

Due to a result by Szegö [24], we find that, for the set E considered in this paper,

$$C(E) = \exp[-V], \tag{10.29}$$

where V is of the form given in (10.8). (For a proof of the equivalence of the various expressions for the transfinite diameter, the reader is referred to [27, theorem 26, p73].) Thus, when $E = [-1, \alpha_1] \cup_{i=1}^{g-1} [\beta_i, \alpha_{i+1}] \cup [\beta_g, 1]$, the transfinite diameter is given by

$$C(E) = \exp\left[\int_1^\infty \left(\frac{1}{t} - \frac{t^g + k_{g-1}t^{g-1} + \cdots + k_0}{\sqrt{(t^2-1)\prod_{j=1}^g (t-\alpha_j)(t-\beta_j)}}\right) \mathrm{d}t\right]. \tag{10.30}$$

This fact is well known and can be found in [30, p 226] and [3] among others.

We proceed by presenting some lemmas that will be required in order to show that extremal polynomials associated with the set E tend towards P_n in the limit as $n \to \infty$:

Lemma 10.1. *Since the B period matrix has the property that* $\mathrm{Re}\{B_{jk}\} = 0$, $j, k = 1, \ldots, g$, *for $\boldsymbol{u} = \mathrm{i}\hat{\boldsymbol{u}}$, with $\hat{\boldsymbol{u}} \in \mathbb{R}^g$ it follows that*

$$\vartheta(\boldsymbol{u}; B) > 1. \tag{10.31}$$

Proof.

$$\begin{aligned}
\vartheta(\boldsymbol{u}; B) &= \sum_{\boldsymbol{s}\in\mathbb{Z}^g} \exp(\mathrm{i}\pi[(\boldsymbol{s}, B\boldsymbol{s}) + 2(\boldsymbol{s}, \boldsymbol{u})]) \\
&= 1 + \left(\sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1\geqslant 1}} + \sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1=0\\ s_2\geqslant 1}} + \cdots + \sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1=s_2=\cdots=s_{g-1}=0\\ s_g\geqslant 1}}\right) \mathrm{e}^{\pi\mathrm{i}(\boldsymbol{s},B\boldsymbol{s})}(\mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{s},\boldsymbol{u})} + \mathrm{e}^{-2\pi\mathrm{i}(\boldsymbol{s},\boldsymbol{u})}) \\
&= 1 + 2\left(\sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1\geqslant 1}} + \sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1=0\\ s_2\geqslant 1}} + \cdots + \sum_{\substack{\boldsymbol{s}\in\mathbb{Z}^g\\ s_1=s_2=\cdots=s_{g-1}=0\\ s_g\geqslant 1}}\right) \mathrm{e}^{\pi\mathrm{i}(\boldsymbol{s},B\boldsymbol{s})} \cosh 2\pi(\boldsymbol{s}, \hat{\boldsymbol{u}})) > 1.
\end{aligned}$$

□

Lemma 10.2. *The vector $\boldsymbol{u}_{\alpha_j}$ is given by the formula*

$$\boldsymbol{u}_{\alpha_j} = \frac{1}{2}\sum_{k=1}^{j}\int_{b_k} \mathrm{d}\boldsymbol{\omega} + \frac{\boldsymbol{e}_j}{2}, \tag{10.32}$$

where $(\boldsymbol{e}_j)_k = \delta_{jk}$. It then follows that

$$\boldsymbol{D} = 2\sum_{j=1}^{g}\boldsymbol{u}_{\alpha_j} = B\begin{pmatrix} g \\ g-1 \\ \vdots \\ 1\end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1\end{pmatrix} \tag{10.33}$$

Proof. We have for $j = 1, \dots, g$

$$\boldsymbol{u}_{\alpha_j} = \begin{cases} \displaystyle\int_{\mathfrak{p}_1}^{\mathfrak{p}_{\beta_g}} \mathrm{d}\boldsymbol{\omega} + \sum_{k=0}^{g-j}\int_{\mathfrak{p}_{\beta_{g-k}}}^{\mathfrak{p}_{\alpha_{g-k}}} \mathrm{d}\boldsymbol{\omega} + \sum_{k=0}^{g-j-1}\int_{\mathfrak{p}_{\alpha_{g-k}}}^{\mathfrak{p}_{\beta_{g-k-1}}} \mathrm{d}\boldsymbol{\omega} & \text{if } j < g \\ \displaystyle\int_{\mathfrak{p}_1}^{\mathfrak{p}_{\beta_g}} \mathrm{d}\boldsymbol{\omega} + \int_{\mathfrak{p}_{\beta_g}}^{\mathfrak{p}_{\alpha_g}} \mathrm{d}\boldsymbol{\omega} & \text{if } j = g, \end{cases} \tag{10.34}$$

where all paths of integration lie in the +sheet of $\mathfrak{R}$ and project onto the upper half of the complex plane. Note that for $k = 2, \dots, g$

$$\int_{\mathfrak{p}_{\alpha_k}}^{\mathfrak{p}_{\beta_{k-1}}} \mathrm{d}\boldsymbol{\omega} = -\tfrac{1}{2}\int_{b_k} \mathrm{d}\boldsymbol{\omega} = -\tfrac{1}{2}B\boldsymbol{e}_k.$$

It then follows that

$$\begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix} = \int_E \mathrm{d}\boldsymbol{\omega} = \tfrac{1}{2}\sum_{k=1}^{g}\int_{b_k} \mathrm{d}\boldsymbol{\omega} + \int_{\mathfrak{p}_{\beta_g}}^{\mathfrak{p}_1} \mathrm{d}\boldsymbol{\omega}$$

It is also easily shown that

$$\boldsymbol{e}_k = \int_{a_k} \mathrm{d}\boldsymbol{\omega} = 2\sum_{l=0}^{g-k}\int_{\mathfrak{p}_{\beta_{g-l}}}^{\mathfrak{p}_{\alpha_{g-l}}} \mathrm{d}\boldsymbol{\omega},$$

from which we deduce that

$$\int_{\mathfrak{p}_{\beta_k}}^{\mathfrak{p}_{\alpha_k}} \mathrm{d}\omega_l = \begin{cases} \frac{1}{2} & \text{if } k = l \\ -\frac{1}{2} & \text{if } k = l-1 \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these last few results into (10.34) the lemma follows. □

Lemma 10.3. *Writing $B = \mathrm{i}\tilde{B}$, we then have the following inequality for any $\boldsymbol{u} \in \mathbb{R}^g$ with the property that $|\mathrm{u}_j| < 1$;*

$$|\vartheta(\boldsymbol{u} - \boldsymbol{D}; B)| \geqslant \frac{\exp[-\frac{\pi g}{\upsilon}]}{\sqrt{\det \tilde{B}}}, \tag{10.35}$$

where υ is the smallest eigenvalue of $\tilde{B}$.

Proof. Using (10.33) and the elementary properties of the theta function,

$$|\vartheta(\boldsymbol{u} - \boldsymbol{D}; B)| = \exp[\pi(\boldsymbol{t}_0, \tilde{B}\boldsymbol{t}_0)]|\vartheta(\boldsymbol{u}; B)|,$$

where $\boldsymbol{t}_0 := (g, \dots, 1)^T$. Note that, as $\tilde{B}$ is positive definite,

$$(\boldsymbol{t}_0, \tilde{B}\boldsymbol{t}_0) \geqslant \upsilon(\boldsymbol{t}_0, \boldsymbol{t}_0) \geqslant 0.$$

Observe the following specialized modular transformation of the theta function known as the Jacobi imaginary transformation [7, ch 2]:

$$\vartheta(\boldsymbol{u};\, B) = \frac{\mathrm{i}^{g/2}}{\sqrt{\det B}} \exp[-\pi \mathrm{i}(\boldsymbol{u},\, B^{-1}\boldsymbol{u})]\vartheta(B^{-1}\boldsymbol{u};\, -B^{-1}).$$

Using this result we find that

$$|\vartheta(\boldsymbol{u};\, B)| = \frac{\exp[-\pi(\boldsymbol{u},\, \tilde{B}^{-1}\boldsymbol{u})]}{\sqrt{\det \tilde{B}}}|\vartheta(-\mathrm{i}\tilde{B}^{-1}\boldsymbol{u};\, \mathrm{i}\tilde{B}^{-1})|.$$

Since

$$(\boldsymbol{u},\, \tilde{B}^{-1}\boldsymbol{u}) \leqslant \frac{(\boldsymbol{u},\, \boldsymbol{u})}{\upsilon} \leqslant \frac{g}{\upsilon},$$

we obtain

$$|\vartheta(\boldsymbol{u} - \boldsymbol{D};\, B)| \geqslant \frac{\exp[-\frac{\pi g}{\upsilon}]}{\sqrt{\det \tilde{B}}}|\vartheta(-\mathrm{i}\tilde{B}^{-1}\boldsymbol{u};\, \mathrm{i}\tilde{B}^{-1})|.$$

The result then follows by applying lemma 10.1. □

Lemma 10.4. *Again writing $B = \mathrm{i}\tilde{B}$, for any $\boldsymbol{u} \in \mathbb{R}^g$,*

$$|\vartheta(\boldsymbol{u} - \boldsymbol{D};\, B)| \leqslant \exp[\pi g^3 \tilde{\upsilon}]\vartheta(0;\, B), \tag{10.36}$$

where $\tilde{\upsilon}$ is the greatest eigenvalue of $\tilde{B}$.

Proof. Using the quasi-periodic property of the theta function and lemma 10.2, we find

$$\vartheta(\boldsymbol{u} - \boldsymbol{D};\, B) = \exp[2\pi \mathrm{i}(\boldsymbol{u},\, \boldsymbol{t}_0)]\exp[\pi(\boldsymbol{t}_0,\, \tilde{B}\boldsymbol{t}_0)]\vartheta(\boldsymbol{u};\, B),$$

where $\boldsymbol{t}_0$ is defined as above, and hence

$$|\vartheta(\boldsymbol{u} - \boldsymbol{D};\, B)| = \exp[\pi(\boldsymbol{t}_0,\, \tilde{B}\boldsymbol{t}_0)]|\vartheta(\boldsymbol{u};\, B)|.$$

Note that, as $\tilde{B}$ is positive definite,

$$0 < (\boldsymbol{t}_0,\, \tilde{B}\boldsymbol{t}_0) \leqslant \tilde{\upsilon}(\boldsymbol{t}_0,\, \boldsymbol{t}_0) \leqslant \tilde{\upsilon} g^3,$$

and from the definition for the theta function

$$|\vartheta(\boldsymbol{u};\, B)| \leqslant \sum_{\boldsymbol{s} \in \mathbb{Z}^g} \exp[-\pi(\boldsymbol{s},\, \tilde{B}\boldsymbol{s})] = \vartheta(\boldsymbol{0};\, B),$$

concluding the proof. □

We now have sufficient information to prove the following theorem:

Theorem 10.5. *The following limit holds:*

$$\lim_{n\to\infty} (\max_{x\in E} |P_n(x)|)^{1/n} = C(E). \tag{10.37}$$

Consequently we find that asymptotically the extremal polynomials for the set E tend towards $P_n(x)$.

Proof. If we can show that, for all $x \in E$,

$$|P_n(x)| \leqslant N(n)C^n(E), \tag{10.38}$$

where $N(n)$ is a positive function of n with the property that $N^{\frac{1}{n}} \to 1$ as $n \to \infty$, then it is certainly true that

$$\lim_{n\to\infty} (\max_{x\in E} |P_n(x)|)^{1/n} \leqslant C(E). \tag{10.39}$$

Equality must then follow by the properties of the transfinite diameter.

By virtue of the expression for $P_n(x)$ in (7.9) it is easily shown that for $x \in [-1, \alpha_1 - \varepsilon] \cup_{j=1}^{g-1} [\beta_j, \alpha_{j+1} - \varepsilon] \cup [\beta_g, 1]$, where $\varepsilon > 0$ is arbitrarily small but fixed, that

$$|P_n(x)| \leqslant N_1(\varepsilon) P_n(1),$$

where $N_1(\varepsilon)$ is a positive real number independent of n. This follows from the fact that $\Psi_n(x)$ is real and $\gamma_j(n)$, $j = 1, \ldots, g$, are bounded in n. Now from the expression for $P_n(x)$ in (5.21) it immediately follows that

$$P_n(1) \leqslant \mathrm{e}^{-n\chi_0}\left(\left| \frac{\vartheta(\boldsymbol{u}^- - \boldsymbol{D})\vartheta(n\hat{\boldsymbol{B}} - \boldsymbol{D})}{\vartheta(\boldsymbol{D})\vartheta(\boldsymbol{u}^- + n\hat{\boldsymbol{B}} - \boldsymbol{D})} \right| + \left| \frac{\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})\vartheta(n\hat{\boldsymbol{B}} + \boldsymbol{D})}{\vartheta(\boldsymbol{D})\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})} \right| \right). \tag{10.40}$$

Note that $\hat{\boldsymbol{B}}$ is a vector with real components. This follows directly from its definition in (4.17) and the fact (10.13) that relates $\mathrm{d}\Omega$ to the equilibrium density of E. Indeed we uncover a physical interpretation of the elements of $\hat{\boldsymbol{B}}$, finding that $-\hat{\mathrm{B}}_j$ is, in fact, equal to the proportion of charges lying on the interval $[\beta_{j-1}, \alpha_j]$, where $\beta_0 := -1$. Thus for any $\boldsymbol{u}$,

$$\vartheta(n\hat{\boldsymbol{B}} + \boldsymbol{u}) = \vartheta(\{n\hat{\boldsymbol{B}}\} + \boldsymbol{u}).$$

It also follows from the fact that the components of $\mathrm{d}\boldsymbol{\omega}$ are real valued on the interval $[1, \infty)$ that the vectors $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are elements of $\mathbb{R}^g$ with components such that $\mathrm{u}_j^\pm \in [0, 1)$, $j = 1, \ldots, g$. Consequently it is easily shown by (10.35) and (10.36) that

$$|P_n(1)| \leqslant N_2 \mathrm{e}^{-n\chi_0},$$

where N_2 is a positive constant independent of n. Recalling that $\chi_0 = V$ and that $C(E) = \mathrm{e}^{-V}$, we see that for all values of n and $x \in [-1, \alpha_1) \cup_{j=1}^{g-1} [\beta_j, \alpha_{j+1}) \cup [\beta_g, 1]$, there exists a positive number N such that

$$|P_n(x)| \leqslant N(\varepsilon) C^n(E). \tag{10.41}$$

It remains to consider the behaviour of $|P_n(\alpha_j)|$, $j = 1, \ldots, g$. We can obtain expressions for $P_n(\alpha_j)$ from (5.21) by expanding both $\mathcal{E}_n(\mathfrak{p}_x)$ and $\tilde{\mathcal{E}}_n(\mathfrak{p}_x)$ around the point $\mathfrak{p}_{\alpha_j}$. Without loss of generality we assume that $\mathfrak{p}_x$ is a point on the +sheet of $\mathfrak{R}$ that corresponds to the real value $x = \alpha_j + \delta$, with $0 < \delta \ll 1$, and any integration contours joining $\mathfrak{p}_1$ and $\mathfrak{p}_x$ are smooth non-self-intersecting and lie in the upper half of the +sheet. Writing $\boldsymbol{u}_x = \boldsymbol{u}_{\alpha_j} + \Delta\boldsymbol{u}$,

$$\begin{aligned}\Delta \mathrm{u}_l = \int_{\mathfrak{p}_{\alpha_j}}^{\mathfrak{p}_x} \mathrm{d}\omega_l &= 2\left(\frac{\sum_{k=1}^{g} (A^{-1})_{lk} \alpha_j^{g-k}}{\sqrt{(\alpha_j^2 - 1)(\alpha_j - \beta_j) \prod_{1 \leqslant m \neq j \leqslant g} [(\alpha_j - \alpha_m)(\alpha_j - \beta_m)]}} \right) \sqrt{\delta} + \mathrm{O}(\delta^{3/2}) \\ &=: 2L_l \sqrt{\delta} + \mathrm{O}(\delta^{3/2}),\end{aligned} \tag{10.42}$$

where this result follows directly from the local expansion of the expression for $\mathrm{d}\omega_l$ given in (4.2). Regarding $\Omega(\mathfrak{p}_x)$, it is easily shown by use of the normalization conditions given in (4.16) that

$$\Omega(\mathfrak{p}_{\alpha_j}) = \int_{\mathfrak{p}_1}^{\mathfrak{p}_{\alpha_j}} \mathrm{d}\Omega = \pi \mathrm{i} \left(1 + \sum_{l=1}^{j} \hat{\mathrm{B}}_l \right). \tag{10.43}$$

By expanding $\mathrm{d}\Omega$ about $\mathfrak{p}_{\alpha_j}$

$$\begin{aligned}\int_{\mathfrak{p}_{\alpha_j}}^{\mathfrak{p}_x} \mathrm{d}\Omega &= 2\left(\frac{\alpha_j^g + \sum_{l=0}^{g-1} k_l \alpha_j^l}{\sqrt{(\alpha_j^2 - 1)(\alpha_j - \beta_j) \prod_{1 \leqslant m \neq j \leqslant g} [(\alpha_j - \alpha_m)(\alpha_j - \beta_m)]}} \right) \sqrt{\delta} + \mathrm{O}(\delta^{3/2}) \\ &=: 2M\sqrt{\delta} + \mathrm{O}(\delta^{3/2}).\end{aligned} \tag{10.44}$$

Using (10.42)–(10.44), from (5.21) we obtain an expansion of of $P_n(\alpha_j+\delta)$ in terms of powers of δ. Setting $\delta=0$ we then find that

$$P_n(\alpha_j) = \frac{\mathrm{e}^{-n\chi_0}}{\sum_{k=1}^{g}\vartheta_k'(\boldsymbol{u}_{\alpha_j}-\boldsymbol{D})L_k}\left[\frac{\mathrm{e}^{-n\Omega(\mathfrak{p}_{\alpha_j})}\vartheta(\boldsymbol{u}^- - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^- + n\hat{\boldsymbol{B}} - \boldsymbol{D})}\right.$$
$$\times\left(\sum_{l=1}^{g}\vartheta_l'(\boldsymbol{u}_{\alpha_j}+n\hat{\boldsymbol{B}}-\boldsymbol{D})L_l - nM\vartheta(\boldsymbol{u}_{\alpha_j}+n\hat{\boldsymbol{B}}-\boldsymbol{D})\right)$$
$$\left. + \frac{\mathrm{e}^{n\Omega(\mathfrak{p}_{\alpha_j})}\vartheta(\boldsymbol{u}^+ - \boldsymbol{D})}{\vartheta(\boldsymbol{u}^+ - n\hat{\boldsymbol{B}} - \boldsymbol{D})}\left(\sum_{l=1}^{g}\vartheta_l'(\boldsymbol{u}_{\alpha_j}-n\hat{\boldsymbol{B}}-\boldsymbol{D})L_l + nM\vartheta(\boldsymbol{u}_{\alpha_j}-n\hat{\boldsymbol{B}}-\boldsymbol{D})\right)\right]. \tag{10.45}$$

Note in particular that $\Omega(\mathfrak{p}_{\alpha_j})$ is imaginary and that by construction $|\sum_{k=1}^{g}\vartheta'(\boldsymbol{u}_{\alpha_j}-\boldsymbol{D}|>0$ as both $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ have at most simple poles with respect to the local parameter at $\mathfrak{p}_{\alpha_j}$. Also observe that

$$\vartheta(\boldsymbol{u}_{\alpha_j}\pm n\hat{\boldsymbol{B}}-\boldsymbol{D}) = \vartheta(\boldsymbol{u}_{\alpha_j}\pm\{n\hat{\boldsymbol{B}}\}-\boldsymbol{D})$$
$$\vartheta_l'(\boldsymbol{u}_{\alpha_j}\pm n\hat{\boldsymbol{B}}-\boldsymbol{D}) = \vartheta_l'(\boldsymbol{u}_{\alpha_j}\pm\{n\hat{\boldsymbol{B}}\}-\boldsymbol{D}), \qquad l=1,\ldots,g.$$

Now, since the theta function is an entire function of g complex variables and consequently its derivative is entire too [6, p 247–248], we conclude that both of these functions are bounded above by a positive constant that is independent of n. Applying (10.35) and (10.36), it then follows that for all values of n there exists a positive constant N_3 such that

$$|P_n(\alpha_j)| \leqslant nN_3\mathrm{e}^{-n\chi_0}.$$

Since $\lim_{n\to\infty} n^{1/n}=1$, identifying $C(E)$ with $\mathrm{e}^{-\chi_0}$, we see that for all values of $x\in E$, (10.38) is satisfied, thus concluding the proof. □

Returning to the specific case where $g=1$, we find an explicit expression for $C(E)$. Equation (10.24) is now

$$V = \int_1^\infty\left(\frac{t+k_0}{\sqrt{(t^2-1)(t-\alpha)(t-\beta)}} - \frac{1}{t}\right)\mathrm{d}t.$$

Standard results for elliptic integrals [12] then give

$$V = \lim_{\nu\to\infty}\left\{\frac{2((1-\beta)\Pi\left(\psi(\nu),\frac{2}{1+\beta},k\right)+(\beta+k_0)F(\psi(\nu),k))}{\sqrt{(1-\alpha)(1+\beta)}} - \ln\nu\right\},$$

where

$$\sin\psi(\nu) = \sqrt{\frac{(\beta+1)(\nu-1)}{2(\nu-\beta)}}.$$

In order to consider the limit above we make use of the following identity [1, p 599], which states that for $l>1$

$$\Pi(\psi,l,k) = -\Pi(\psi,m,k)+F(\psi,k)+\frac{1}{2\chi}\ln\left|\frac{\Delta(\psi)+\chi\tan\psi}{\Delta(\psi)+\chi\tan\psi}\right|, \tag{10.46}$$

with

$$\begin{aligned} m &:= \frac{k^2}{l} \\ \Delta(\psi) &:= \sqrt{1-k^2\sin^2\psi} \\ \chi &:= \sqrt{(1-m)(l-1)}. \end{aligned} \tag{10.47}$$

By expanding these quantities in terms of $\frac{1}{\nu}$, the limit is

$$V = \frac{2(1+k_0)}{\sqrt{(1-\alpha)(1+\beta)}}F(\phi,k) + \frac{2(\beta-1)}{\sqrt{(1-\alpha)(1+\beta)}}\Pi\left(\phi,\frac{\beta-\alpha}{1-\alpha},k\right) - \ln\frac{\beta-\alpha+2}{4}.$$

In order to exploit this result, we require an alternative representation in terms of theta functions. Using (10.17), again we set $r = F(\phi,k)$, but this time $u = \mathrm{sn}^{-1}\sqrt{(1+\beta)/2} = F(\phi,k)$, thus

$$\Pi\left(\phi,\frac{\beta-\alpha}{1-\alpha},k\right) = \frac{\sqrt{(1-\alpha)(1+\beta)}}{\beta-1}\left[\frac{1}{2}\ln\frac{\vartheta_4(2\mathrm{u}^+)}{\vartheta_4(0)} - \mathrm{u}^+\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)}\right],$$

having observed that $r/2K = \mathrm{u}^+$. Recalling that

$$k_0 = \frac{2}{A}\frac{\vartheta_1'(\mathrm{u}^+)}{\vartheta_1(\mathrm{u}^+)} - 1,$$

we simplify the expression for the conductor potential substantially, obtaining

$$V = \ln\left[\frac{4\vartheta_4(2\mathrm{u}^+)}{(\beta-\alpha+2)\vartheta_4(0)}\right].$$

This can be further manipulated by using the following identities [12], [28, ch 9]:

$$\begin{aligned}
\vartheta_1\left(\frac{u}{K}\right) &= \sqrt{k}\vartheta_4\left(\frac{u}{K}\right)\mathrm{sn}2u\\
\vartheta_1'(0) &= \pi\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\\
\vartheta_2(0) &= \sqrt{\frac{2kK}{\pi}}\\
\vartheta_3(0) &= \sqrt{\frac{2K}{\pi}}.
\end{aligned} \tag{10.48}$$

Combining these allows us to write

$$V = \ln\left[\frac{\vartheta_1(2\mathrm{u}^+)}{\vartheta_1'(0)}\frac{2K}{(\beta-\alpha+2)\mathrm{sn}2r}\right].$$

This can be simplified by employing the fact that [28, ch 10]

$$\mathrm{sn}2u = \frac{2\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}{1-k^2\mathrm{sn}^4u}. \tag{10.49}$$

Since $\mathrm{sn}r = \sqrt{(1+\beta)/2}$, $\mathrm{cn}r$ and $\mathrm{dn}r$ then follow from their respective definitions given in (10.19), we then obtain

$$\mathrm{sn}2r = \frac{2\sqrt{(1-\alpha)(1+\beta)}}{(\beta-\alpha+2)},$$

and finally arrive at the desired representation:

$$V = \ln\left[\frac{A\vartheta_1(2\mathrm{u}^+)}{\vartheta_1'(0)}\right]. \tag{10.50}$$

Hence in the case where E is composed of the two disjoint intervals of the real line, $[-1,\alpha]$ and $[\beta,1]$, the transfinite diameter is identified as

$$C([-1,\alpha]\cup[\beta,1]) = \frac{\vartheta_1'(0)}{A\vartheta_1(2\mathrm{u}^+)}. \tag{10.51}$$

Note that an equivalent expression for this quantity was derived by Akhiezer in his consideration of Solotareff's problem [5, p 288].

We can now examine the behaviour of $\sigma_n(x)/n$ of the genus 1 polynomial P_n, as $n \to \infty$, for x ranging over the set E. Recall that

$$\sigma_n(x) = \frac{p(x)}{h_{n-1}}[P_n(x)P_{n-1}(x)(f_1(x;n) - f_1(x;n-1)) + f_2(x;n)Q_n(x)P_{n-1}(x) - f_2(x;n-1)P_n(x)Q_{n-1}(x)],$$

where

$$f_1(x;n) := \frac{1}{2}\left(\frac{1}{x-\gamma(n)} - \frac{1}{x-\alpha}\right) \qquad f_2(x;n) := \frac{nx - c_0(n) - \frac{y_1}{2(x-\gamma(n))}}{x-\alpha}. \tag{10.52}$$

Using (10.51) and (8.16) we uncover an expression for the L_2 norm in terms of the transfinite diameter:

$$h_n = 2[C(E)]^{2n}\frac{\vartheta_3((2n+1)\mathrm{u}^+)}{\vartheta_3((2n-1)\mathrm{u}^+)}. \tag{10.53}$$

Noting that for $\vartheta_3(u)$ it has no real zeros, it is then easily shown that, as $n \to \infty$,

$$h_n \sim [C(E)]^{2n}. \tag{10.54}$$

Restricting x to the interval $[1, \alpha) \cup [\beta, 1]$, it follows from (10.41) that asymptotically

$$\frac{P_n(x)P_{n-1}(x)}{h_{n-1}} = \mathrm{O}(1) \tag{10.55}$$

and from (8.33) and (10.22), that

$$c_0(n) = -nk_0 + \mathrm{O}(1)$$

as $n \to \infty$. Therefore in this limit

$$f_2(x;n) = \frac{n(x+k_0)}{x-\alpha} + \mathrm{O}(1). \tag{10.56}$$

Recalling (1.9), we see that for $x \in [-1, \alpha) \cup [\beta, 1]$

$$\sigma_n(x) \simeq \frac{\mathrm{i}n(x+k_0)}{\pi\sqrt{(x^2-1)(x-\alpha)(x-\beta)}}, \qquad n \to \infty,$$

that is

$$\lim_{n\to\infty}\frac{\sigma_n(x)}{n} = \sigma^*(x). \tag{10.57}$$

Thus the equivalence of the equilibrium density for the set E and the limiting expression for $\sigma_n(x)/n$, for x an interior point of E, has been established for the genus 1 case. This is the 'scaling' limit referred to in section 1.

It is important to note the restriction on values of x. In the limiting form above, the appearance of $\sqrt{x-\alpha}$ in the denominator is due to the fact that

$$P_n'(x) = f_1(x;n)P_n(x) + f_2(x;n)Q_n(x),$$

where both f_1 and f_2 have poles at α. Clearly as P_n' is a polynomial of degree $n-1$, it has no poles at any finite value of x. Thus if we were to write the expression above as a single rational function the polynomial in the numerator would be identically zero when $x = \alpha$. Thus referring to (10.1), we see that $\sigma_n(\alpha) = 0$ for all n. However, stipulating that $x \in [-1, \alpha) \cup [\beta, 1]$, the expressions for f_1 and f_2 are bounded in x and the asymptotic analysis proceeds as above. The special nature of the point at α is clearly illustrated in figure 11. Here $\sigma^*(x)$ is plotted

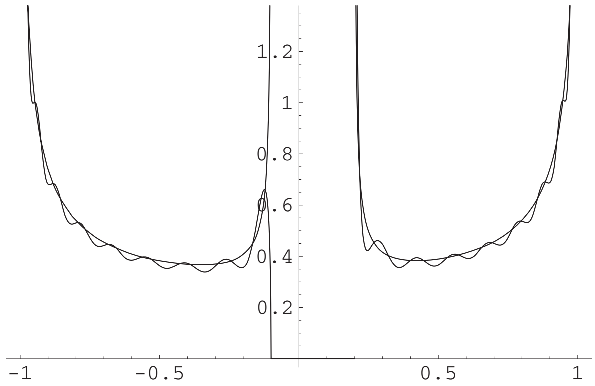

Figure 11. The equilibrium distribution $\sigma^*(x)$, contrasted with $\sigma_n(x)/n$, where $\alpha = -0.1$, $\beta = 0.2$ and $n = 20$.

together with $\sigma_n(x)/n$, with $n = 20$, $\alpha = -0.1$ and $\beta = 0.2$, for $x \in E$. Note then the good agreement at all points, apart from those in the immediate vicinity of α, where $\sigma^*(x)$ is shown to diverge and $\sigma_n(x)/n \to 0$.

Concerning the genus $g \geqslant 1$ case, using the expression for h_n given in (5.13) instead of (8.16), it still follows that

$$h_n \sim [C(E)]^{2n}, \qquad n \to \infty.$$

Thus if we can prove that as $n \to \infty$, $c_j(n) = n\check{c}_j + \mathrm{O}(1)$, where $\check{c}_j$ is independent n, the genus 1 analysis can be repeated and

$$\lim_{n\to\infty} \frac{\sigma_n(x)}{n} = \frac{\mathrm{i}\left(x^g - \sum_{j=0}^{g-1} \check{c}_j x^j\right)}{\pi\sqrt{(x^2-1)\prod_{j=1}^{g}(x-\alpha_j)(x-\beta_j)}}, \qquad x \in E\backslash\{\alpha_j : j = 1, \ldots, g\}. \tag{10.58}$$

In fact we have

Theorem 10.6.

$$\lim_{n\to\infty} \frac{c_j}{n} = -k_j, \qquad j = 0, 1, \ldots, g-1. \tag{10.59}$$

Proof. Using the expression of $\boldsymbol{D}$ and the quasi-periodicity of the theta function we can simplify (5.21) and (5.22). Computing the derivative of $P_n(x)$ with respect to x using the simplified forms gives

$$P_n'(x) = n\Omega'(x)R_n(x) + C^n(E)\left[\mathrm{e}^{-n\Omega(x)}\vartheta_-(n)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\vartheta(\boldsymbol{u}_x + \{n\hat{\boldsymbol{B}}\})}{\vartheta(\boldsymbol{u}_x)} + \mathrm{e}^{n\Omega(x)}\vartheta_+(n)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\vartheta(\boldsymbol{u}_x - \{n\hat{\boldsymbol{B}}\})}{\vartheta(\boldsymbol{u}_x)}\right],$$

where

$$\vartheta_\mp(n) := \frac{\vartheta(\boldsymbol{u}^\mp)}{\vartheta(\boldsymbol{u}^\mp \pm \{n\hat{\boldsymbol{B}}\})}.$$

As noted previously $\vartheta_\pm(n)$ is bounded in n and, for $x \in E$, $\Omega(x)$ is pure imaginary. Now since $\vartheta(\boldsymbol{u}_x \pm \{n\hat{\boldsymbol{B}}\})$ is an entire function of g complex variables which in turn implies its derivative is also an entire function, we conclude that, for any x fixed in E, $\frac{\mathrm{d}}{\mathrm{d}x}\vartheta(\boldsymbol{u}_x \pm \{n\hat{\boldsymbol{B}}\})$ is bounded (in n). Furthermore, according to a theorem from potential theory, for any closed bounded sets S_1 and S_2 with $S_1 \subset S_2$, $C(S_1) \leqslant C(S_2)$ [27, theorem 3.3, p 56]. Therefore taking S_1 to be our E and S_2 to be $[-1, 1]$, we have $C(E) \leqslant C([-1, 1]) = \frac{1}{2}$.

Thus for x fixed in E

$$P_n'(x) = n\Omega'(x)R_n(x) + \mathrm{O}(1).$$

Comparing this with the first of the differential relations in (6.40), we find

$$w(x) \lim_{n\to\infty} \frac{f_2(x)}{n} = \Omega'(x), \tag{10.60}$$

for almost every $x \in E$. Noting the form of $f_2(x)$ given by (6.34) and the facts that γ_j and y_j are bounded in n, $P_n(x)$ is seen to be bounded for x in E (in fact exponentially small in n, since the transfinite diameter, $C(E)$, is less than $\frac{1}{2}$) and $-\mathrm{i}\pi\sigma^*(x) = \Omega'(x)$, we see that

$$\lim_{n\to\infty} \frac{c_j(n)}{n} = -k_j, \qquad j = 0, 1, \ldots, g-1. \tag{10.61}$$

□

We conclude this paper with the following remark. It is well known that the problem of orthogonal polynomials can be equivalently reformulated as a matrix Riemann–Hilbert problem [10]. Such a formulation will enable one to deduce, for example, non-linear equations satisfied by a_n, b_n, and those partial differential equations satisfied by them associated with the change of the end points of the interval. Work is presently underway in this direction and the results will be published in a separate paper.

Acknowledgment

The authors should like to thank Professor Alexander Its for illuminating discussions.

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